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**ABSTRACT**

of the dissertation for the degree of Doctor of Science

**DIRECT AND INVERSE PROBLEMS FOR NON-SELF-ADJOINT HILL OPERATOR**

Speciality: 1211.01 –Differential equation

Field of science: Mathematics

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**GENERAL CHARACTERISTICS OF THE WORK**

**Rationale of the topic and the degree of development**. This dissertation is devoted to solve direct and inverse problems for Schrödinger operators with complex-valued, periodic potentials in various settings.

In various branches of mathematical physics, in particular, in non-Hermitian quantum mechanics and crystal theory, it becomes necessary to study differential operators with complex-valued and periodic coefficients.

The main idea of ​​non-Hermitian quantum mechanics is that quantities measured in an experiment are always described by real numbers, not complex numbers, therefore, each observable quantity is associated with an operator acting in the space of state vectors, the eigenvalues ​​of which are the measurement results. Our requirement in constructing operators is that their eigenvalues ​​must be real. The implementation of non-Hermitian quantum mechanics consists of replacing the Hermitian conjugation by a PT- symmetry transfor-mation. The P-symmetry transformation (reflection of spatial coordinates) consists, for example, in changing the sign in front of the coordinate operator, and the T-symmetry transformation (time reversal) consists in changing the sign of the impulse (but not the coordinate), as well as replacing  by . The potentials considered in the thesis are of the form



where certain conditions are fulfilled for numbers  and are PT-symmetric for , i.e. ... . These potentials were first considered in M.G.Gasymov in various formulations for the Schrödinger operator and was investigated by I.M.Guseinov, A.Orudzheva, E.Orudzheva,V. Guillemin and A.Uribe, R.Carlson,

K.Shin, L.Pastur, V.Tkachenko and Fegan.

              In the dissertation work, three main areas are considered.

* We study inverse problems for a pencil of differential operators, as well as for higher-order operators with a complex-valued periodic potential. These cases are very complex. It turned out that the Goursat-type problem for transformation operators is ill-posed and has a solution in the general case only for equations with analytic coefficients. In this regard, it is of great interest to choose some classes of ordinary differential operators for which transformation operators exist.
* We study inverse problems for various types of discontinuous differential operators with complex-valued periodic potential. In connection with important applications in quantum physics, it is of interest to study the spectral characteristics of discontinuous differential operators. As a rule, the problems under consideration are related to the discontinuous properties of the physical characteristics of the medium.
* Scattering problems on quantum graphs are studied. Hill's problem with a complex-valued potential on quantum graphs arises in the most diverse problems of natural science - from waveguide systems and neural networks to discrete-continuous approximations of the Laplacian on a Riemannian manifold.

**Goal and tasks of the research.**

* Solution of a characterization problem, i.e., a complete inverse problem for a pencil of differential operators.
* Investigation of spectral characteristics, as well as solution of the inverse problem for high-order differential equations with a spectral parameter polynomially included in the equations.
* Investigation of the indefinite spectral problem for the Hill operator from two spectra.
* Solution of the inverse problem for the wave equation with discontinuous velocity.
* Study of wave propagation in a non-conservative medium by studying the spectral characteristics of a beam of differential operators with a delta-shaped potential.
* Study of wave propagation on branching strings by studying the spectral characteristics of the Schrödinger operator on quantum graphs.

**Investigation methods.**  The dissertation work uses the methods of the spectral theory of operators, the theory of functions of a complex variable, the theory of differential equations and equations of mathematical physics.

              **Key points of the dissertation which will be defended**:

* complete inverse problems for a pencil of differential operators
* inverse problems for high-order differential equations with spectral parameter polynomially included in the equations
* inverse problems for different types of discontinuous differential operators with complex-valued periodic potential
* scattering problems on quantum graphs

**Scientific novelty of the research.** The following main results were obtained.

* Necessary and sufficient conditions are obtained for a given sequence of complex numbers to be a set of spectral data for a second-order operator pencil and higher-order differential operators.
* The direct and inverse problems of spectral analysis for  order ordinary differential equations with polynomially depending on the spectral parameter are solved. It is shown that the continuous spectrum of the operator pencil fills out the rays , where  , and on the continuous spectrum there are spectral singularities that coincide with numbers of the form  By generalized normalization numbers, the inverse problem of recovering the coefficients is solved.
* The inverse problem for the Schrödinger operator with complex-valued periodic potentials and a discontinuous coefficient on the whole real axis is solved. The main characteristics of fundamental solutions are investigated, the spectrum of the operator is studied. The inverse problem is formulated, the uniqueness theorem is proved, and a constructive procedure for solving the inverse problem is proposed
* The classical Hill problem with complex potential is extended to star and loop graphs. The definition of the Hill operator on such graphs is given. The operator is defined by complex, periodic potentials and special boundary conditions are used to connect the values ​​of the functions at the vertices. An effective description of the form of the resolvent is given, and the spectrum is accurately described, and the inverse problem for reflection coefficients is solved.

**Theoretical and practical value of the study.**

Mathematical methods are given for the study of various direct and inverse problems for the Hill operator with a complex, periodic potential. For the first time, the complete solution of the inverse problem for a pencil of differential operators with complex periodic potentials is investigated, the inverse problem for normalization numbers for a high-order operator with a polynomially depending spectral parameter is solved. Various discontinuous inverse problems are investigated. The results obtained for the classical Hill problem are generalized to quantum graphs. The obtained results of the dissertation can be applied in the theory of direct and inverse problems of mathematical and quantum physics.

**Approbation and application**. The results of the dissertation were presented at the following seminars: at the seminar of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan (acad. F.G.Maksudov); at the seminat of the “Non-harmonic analysis” (corr-member of ANAS, prof.B.T.Bilalov), “Functional analysis” (prof. H.I.Aslanov) və “Differential equation” (prof. A.B.Aliyev), at the department of Applied Mathematics of Baku State University (acad. M.G.Gasimov); the Institute of Applied  Mathematic Baku State University (acad. F.A.Aliyev); at the University of Nantes (prof. A.Nachaoui, France, 2011); at the Polytechnic University of Valencia (prof. Luis García Raffi, Spain, 2016); at a seminar on differential equations of the Masaryk University, Brno, (the Czech Republic, 2017); at Keel University, (prof. Y.Kaplunova, England, 2017, 2019); at the international conference on ill-posed and inverse problems dedicated to the 70th anniversary of academician M.M. Lavrentieva (August 5-9, 2002 Novosibirsk, Russia); at the 38-th annual Iranian Mathematical conference (03-10 September 2007, Zanjan, Iran); Third conference of the World Mathematical Society of Turkic countries (June 30 - July 4, 2009, Baku, Azerbaijan); Bulgarian-Turkish-Ukrainian scientific conference "Mathematical analysis, Differential Equations and their applications" (September 15-20, 2010, Sunny Beach, Bulgaria); International conference on functional analysis dedicated to tthe 90th anniversary of V.E.Lyantse (3-10 November 2010, Lviv, Ukraine); VI congress of the Turkic World Mathematical Society (July 11-13 -2018, Baku, Azerbaijan); VI Congress of the Turkic World Mathematical Society (July 11-13 -2018, Baku, Azerbaijan).

**Personal contribution of the author.** All the results obtained in the dissertation belong to the author.

- The complete inverse problem is solved for operator pencils of the second-order and operators of higher-order

- Spectral problems for discontinuous spectral problems are solved

- Classical results obtained for operators with complex periodic potentials extended to star and loop-shaped graphs

**Publications of the author.** Publications in editions recommended by HAC under the President of the Republic of Azerbaijan -24, conference materials - 11.

**The name of the institution where the thesis is performed** Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

**Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately).** The total volume of the dissertation –401001 signs (the title page-645 and content 3524 signs, introduction – 75000  signs, chapter I – 122000  signs, chapter II – 38000 signs, chapter III – 76000 signs chapter IV-84000 signs, conclusion – 1832 signs).

**THE MAİN CONTENT OF THE DISSERTATION**

        The thesis consists of an introduction, four chapters and a list of used literature.

The introduction justifies the relevance of the research topic and shows the degree of its elaboration, formulates the purpose and task of the research, provides scientific novelty, notes the theoretical and practical value of the research, and also provides information on the approbation of the work.

In the **first chapter** of the dissertation, the problem of characterization is solved, which consists of determining the necessary and sufficient conditions for the scattering data so that the reconstructed potentials belong to a certain class.

Introduced the class  of all complex-valued,  periodic functions on the real axis , belonging to space  and its subclass  , consisting of functions of the form

                                (1)

              The object of research is a non-self-adjoint operator pencil  generated by the differential expression

                  (2)

in a space   with potentials  for which the conditions

                       (3)

- spectral parameter.

              In 1.1.2, the analytic properties of solutions to the equation  are investigated, for which the fundamental system of solutions of the equation

  (4)

is explicitly constructed

**Theorem 1.**Let the potentials  belong to space for which conditions (3) are satisfied. Then the differential equation (4) has particular solutions that can be represented in the form of

  (5)

Here  , are the   numbers for which the series converge

              (6)

     It follows from representation (5) that  is a holomorphic function for  and can have first-order poles at points

              **Lemma 1.**For the  functions form a fundamental system of the solution to the equation (4).

It is shown that with the notation

 (7)

the relation holds

  (8)

Comparison of the formulas for these expressions shows

                                              (9)

              **Definitions 1.**A sequence constructed using formulas (9) is called a set of spectral data of an operator  with poten-tials .

              **Theorem 2. (**themain result of this chapter). To a given sequence of complex numbers to be a set of spectral data of an operator  with potentials,  it is necessary and sufficient, the simultaneous fulfilment of the condition

1.  (10)

2. Infinite determinant

  (11)

exists, is continuous, does not vanish in a closed half-plane , and is analytic inside an open half-plane .

In 1.1.3 the inverse problem of scattering theory is considered. First, the spectral properties of the operator  are investigated, which are based on the study of the Floquet solutions to the equation



Since   analytically continued into the upper half-plane  , then, of course, it is convenient to consider (1) for all  Assuming    we obtain the equation

               (12)

in which

                         (13)                            (14)

              As a result, obtained the equation (12) where the potentials exponentially decrease at . Whence it follows that it is possible to use the method of transformation operators, well-known in the theory of inverse problems.

              According to the definition of the transformation operator to infinity, the solution to equation (12) can be represented in the form of

           (15)

Comparing the formulas (15) and (12), we have

         (16)                     (17)

              The kernel of the transformation operator  and the  in our case are constructed constructively. Thus, we have proved the following theorem.

**Theorem 3.**The function  and the kernel , transformation operator to infinity to the equation (12) allows the representation (16) and (17) respectively, where the series

 :

converge.

              Further, the questions of the solvability of the main equation and the uniqueness of the solution of the inverse problem are studied.

              Section 1.2 solves the complete inverse problem for the high-order operator  generated by the differential expression

  (18)

in space  with potentials , where   consists of functions of

   (19)

and is a subclass of the class of  all  periodic complex-valued functions on the real axis  belonging to space  .

              In this section, we introduce some well-known facts and concepts that are used in solving the problem and also formulate the main result.

**Theorem 4.**For a given sequence of complex numbers to  to be a set of spectral data of an operator  generated by differential expression (18), it is necessary and sufficient to simultaneously satisfy the condition

(1)   (20)

2) infinite determinant

 (21)

exists, is continuous, does not vanish in a closed half-plane,  and is analytic inside an open half-plane.

The direct and inverse problem of spectral analysis for one class of differential equations of even order with a polynomially entering spectral parameter is solved in section 1.3.1.

A differential operator pencil  generated by an expression of the form

 (22)

where

 (23)

and a number

 (24)

converges, but  is a spectral parameter.

              A resolvent is constructed and the spectrum of an operator pencil  is studied  using the theorem

**Theorem 5.**The operator pencil  has a purely continuous spectrum that fills out the rays



and on the continuous spectrum, there may be spectral singularities that coincide with numbers of the form 

Let be





Expansion in eigenfunctions is obtained, and the inverse problem is solved using the following theorem.

**Theorem 6.**  Let for the given numbers   the following conditions be satisfied

; 

              Then there are functions  of the form (23) for which condition (24) is satisfied and the numbers  are “generalized normalization numbers” of the operator with recovered coefficients.

              Section 1.4 solves the problem of determining the Hill operator from two spectra.

Let us consider the differential equation

           ( 25 )

where  is a complex periodic function of the form (1), (3) with the boundary conditions

 (26)

 (27)

The eigenvalues  and  ​​of problems (25), (26) and (25), (27) are also introduced . It turns out that, if   and   solutions of equation (25), satisfies the initial conditions



then the eigenvalues ​​of boundary value problems (25), (26) and (25), (27) coincide, respectively, with the zeros of the functions

  and .

**Theorem 7.**The potential  is uniquely determined by the spectra  and  of the problems (25), (26) and (25), (27), respectively.

**Chapter II** is devoted to the study of indefinite spectral problems for potentials linearly depending on the  so-called energy-dependent potentials, that is, the pencil of  non-self-adjoint differential operators generated by the formal differential expression



in space . Ther**e**  is a complex number, and the coefficients  are defined as (1), (3) and for the  fulfilled

                            (28)

              In 2.2, special solutions of the equation are studied  using the following theorem

**Theorem 8.**Let it   have the form (1)-(3),  res-pectively, and for satisfies condition (28).

Then the equation

          (29)

has a solution of the form

       







where the numbers and  are determined from the recurrence relations and the series

 : 

converge.

              It will be easy to notice that the functions     form a fundamental system of solutions to equation. (29) for   and 

Then, using the conjugation conditions



we find that each solution to equation. (29) can be represented as linear combinations of these solutions

 (30)

 (31)

Using (30), (31), we obtain the  following relations  for the coefficients , ,



Let us denote



      It was found that the functions   and  are linearly dependent. Hence,

                             (32)

Analysis of formula (32) shows that .

              The 2.3 studied the spectrum of  in space .

 Let us divide the plane into sectors



The following representation holds for the kernel of the resolvent of the operator







Directly from the form of the resolvent, one can obtain that for all outside of  and  , the resolvent of the operator exists and is bounded.

**Theorem 9.** Theoperator  has no real and purely imaginary eigenvalues. The continuous spectrum fills out the axes and on which there may be spectral singularity that coincides with numbers of the form  The eigenvalues of the operator are bounded and coincide with the zeros of the functions  on the sectors



 respectively.

**Definition 2**. The set of quantities  is called the spectral data of the operator .

**Theorem 10.**All the numbers  and  can be uniquely determined by using the numbers 

The **third chapter** is devoted to the spectral analysis of differential operators with truncated coefficients.

Section 3.1 consists of four parts, where discontinuous inverse problems of spectral analysis are solved.

The inverse problem of spectral analysis is solved for the wave equation with discontinuous wave propagation.

For this, the differential equation is considered

     (33)

in space  under the assumption that the potential   is determined using (1), (3), and



a   is a complex number.

              In 3.1.2, the main properties of particular solutions of equation (33) are studied.

It is shown that the equation



has solutions of the form





where the numbers are  determined from the recurrence relations

 



for what converged the series

.

Note that the functions  and    and are linearly independent and their Wronskian is equal to .

Then each solution to equation (33) can be represented as a linear combination of these solutions.

 

 

The coefficients are expressed in terms of the Wronskian of solutions   and .

Indeed







Following the physical meaning of the solutions  and , we   will call the coefficient  the amplitude of the reflection coefficient to the right,  and  the amplitudes of the transmission coefficients .

**Lemma 2**. The eigenvalues ​​of the operator  are finite and coincide with the squares of the zeros of functions  from the sectors  respectively.

              It is shown that the operator , generated by    the differential expression  in space , the kernel of the resolvent has the form



when  and



for .

              Thus, there exists and is bounded for all  outside the positive semiaxis and Moreover, it was proved that the coefficient  is an analytic function for  and has a limited number of zeros, moreover, if then



We get that the eigenvalues ​​of the operator  are finite and coincide with the squares of the zeros of the functions  from the sectors  , respectively.  Note that we can obtain a further useful relation



**Theorem 11**. The spectrum of an operator  is real, purely continuous and fills out the semiaxis , while the continuous spectrum may have spectral singularities of the first order, which coincide with numbers of the form 

            The problem of eigenfunction expansion discussed in paragraph 3.1.3, where it is proved that for an arbitrary function  belonging to space  we have the following expansion in eigenfunctions





              In 3.1.4, the inverse problem for an operator  based on spectral data  is solved. It is proved that the spectral data   uniquely determines the function , and the number 

              Section 3.2 considers the problem of the spectrum of a non-self-adjoint operator pencil with a discontinuous coefficient.

             A spectral problem for an operator pencil  with complex periodic potentials in the space generated by the differential expression

  (34)

where - is a complex number, and for the coefficients conditions (1), (3) are satisfied and  has the form

  ( 35 )

where 

It is known that the study of the spectral properties of an operator pencil  is based on the analysis of solutions to the equation

  (36)

  **Theorem 12**. Let the potentials  have the form (1), (3) and condition (35) is satisfied for. Then the equation   has solutions of the form



for    where the numbers   and  are determined from recurrence relations and   admits double term differentiation.

**Theorem 13**. The spectrum of the operator pencil  consists of a continuous spectrum filling the axis  on which there can be spectral singularities that coincide with numbers of the form     and a finite number of eigenvalues.

In 3.3 we study the spectral analysis of one class of non-self-adjoint differential operator pencils with a generalized function.

              An inverse problem is considered for a pencil  of  non-self-adjoint differential operators generated by a formal differential expression



with a generalized function in space . Here  is the Dirac delta function,  is a real number,   is a complex number, and the coefficients  satisfy the conditions (1), (3).

It is proved that the spectrum of the operator pencil consists of a continuous spectrum filling the axis on which there can be spectral singularities that coincide with numbers of the form   and at most one eigenvalue.

The inverse problem is solved, where the problem of determining the functions,  and  is posed,  by numbers . In the beginning, we found clear links between the sequences  and , .



a  is defined using the equality



These relations are the basic equations for determining   and  by the numbers .

**Theorem 14.** For the numbers  to be the “normalization” numbers of an operator pencil of type  with a potential of the form (1), (3), it is sufficient that the conditions

 

where .

Section 3.4 investigates the spectral analysis of a non-self-adjoint Hill operator with a step potential.

Considered the differential equation

                                                  ( 37 )

in space  where the prime denotes the derivative for the spatial coordinate under the assumption that the potential



 and is a complex number.

Then the equation



has a solution of the form



where



and the numbers  are determined from recurrence relations and the series  converges.

Since the functions  and  are linearly independent solutions of equaiton (37), respectively, for   and , then continuing as a solution to the equation using the conjugation conditions



it is easy to see that the following equalities hold

, 

, .

      Let





then

 and 

           In section  3.4.2 proved that the spectrum of the operator consists of a continuous spectrum that fill out the axis  which may be the spectral features that coincide with numbers of the form  and has a finite number of eigenvalues, defined as the root of the equations .

In 3.4.3, the inverse problem is studied.

Inverse problem: Reconstruct a potential  based on the given spectral data .

              In this section, we give a constructive procedure for solving the inverse problem from given spectral data. Thus, from the spectral data , the  potential   can be recovered unambiguously and efficiently.

The purpose of the fourth chapter is the spectral analysis of wave propagation in a layered, inhomogeneous medium, such as a branching tube or a system of connected strings.

Section 4.1 gives the formulation of the direct problem.

To study the propagation of waves on branching strings, we must consider the system of equations



 where the potentials  and  ,  are of the form





with the following system of boundary conditions at the initial points of the positive semiaxis

           (38)

in the space  where the notation with the index  is used to denote the starting point of the  th positive semiaxis, and the direct sum of the spaces is denoted by . The prime means the derivative for the spatial coordinate and    is a complex number.

              For simplicity of obtaining results without loss of generality in the future, we will consider the case .

    On the outside space 



where   with dot product 

and consider the operator 

Where    with domain



for all   



 Then the problem can be interpreted as a study of an operator  on the above non-compact graph. The spectral problem can be described as follows:

Find a function   satisfying the Sturm-Liouville equation

 (39)

on  connected at zero with usual Kirchhoff condition and by the initial conditions for the functions , .

a)  is   continuous at the nodes of the graph, in particular, for our graph

                                             (40)

b) the sum of derivatives over all branches outgoing from a node, calculated for each node, is equal to zero

                  (41)

 For each fixed  on the edge , there is a fundamental system of solutions  to equation ( 39 ), which for     and  has the form:



where   are numbers and for them the following series converge:



As a solution to the problem, we will understand the matrix

 on a non-compact graph based on the following conditions:

 1. 

2.  a solution on the edge 

3. 

and



The coefficients  and can be found by writing down the boundary conditions (40), (41) for the solution . To be specific, suppose , then





We solve these equations for ,  and . Noting that for the Wronskian solutions we get







where



The coefficients  and  can be found by writing down the boundary conditions (61), (62) for the solution 

In 4.2 we study the properties of the spectrum of an operator . It is proved that the operator  has no real eigenvalue.

**Theorem 15**. The eigenvalues ​​of the operator  are finite and coincide with the zeros  of the function



where



**Theorem 16**. The spectrum of the operator  consists of a continuous spectrum filling out the axis  on which there may be spectral singularities that coincide with the numbers of the form  .

              In 4.1.2, we solve the inverse spectral problem on a star graph.

              **Inverse problem**: Taking into account the spectral data, the reflection coefficients   on each  edge, construct and the potentials  and  where 

**Theorem 17**. On each fixed edge ,   , the following relation is fulfilled





Shown, that



These relations are fundamental equations for the recovery and using predetermined integers .

**Theorem 18**. All the numbers  and  can be uniquely determined from the known numbers  

**Theorem 19**. Specification of the spectral data uniquely determines the potentials ,  on each edge 

In 4.2 considered the inverse spectral problem for the Hill operator on a loop graph.

Considered a non-compact graph  where the half-line is attached to a loop. The graph consists of a non-compact part, which is the axis , the compact part of the loop, the  length of which, for the sake of definiteness, is taken equal to  and  which correspond to the attachment point.

We will investigate a spectral problem describing the one-dimensional scattering of a quantum particle on a graph .

          (42)



.

      In (42), differentiability  is understood as differentiability for   if   and differentiability to  , when  . At the vertex of the graph, differentiability is undefined. We assume that the potential is defined as where



with the condition   is a spectral parameter.

The solution to the spectral problem (42) will be found as



on a non-compact graph  provided that the following conditions are satisfied.

1) 

2) 

where   is the reflection coefficient and



in this case, the numbers are  determined from the following recurrence relations



for which the series  conver-ges. Note that in our case the following relation is valid



To construct a solution  on the compact part of the graph, first, we construct the Green's function of the operator on the compact part of the graph on the loop . Green's function on a loop can be constructed using the fundamental solutions  satisfying the following conditions



In this case, the boundary conditions for the Green's function on the loop are the conditions

                 (43)

The function required to construct a solution to the spectral problem on a loop has the form





Thus, we can look for a solution to the spectral problem as follows



where  is constant.

Then using boundary conditions (42) we have





Taking into account conditions (43) for the Green's function on the loop, we have





So



or



and for the reflection coefficient, we have



The main idea of the solution of the inverse problem for the considered system is its reduction to two independent problems of reconstruction of the potentials  at the edges  and  respectively.

Since the coefficients can be found using the matching conditions





at the central vertex, it is natural to formulate the inverse problem – recovering of the potential   on a non-compact graph  from the reflection coefficients, and the set of eigenvalues ​​of the Dirichlet problems



and Neumann problems



**Inverse problem**: Given the spectral data: the set of eigenvalues ​​of the Dirichlet and Neumann problems, as well as the reflection coefficient , reconstruct the potential  on the loop graph.

              **Theorem 20**. Specification of spectral data uniquely determines the potential  on the loop graph.

              In 4.3, we consider the inverse spectral problem for the Dirac operator on a star graph.

              Consider a non-compact graph  with a single vertex, in which a finite number of edge are connected where the notation  is used, to denote the initial point of the j -th positive semiaxis.

              We define space  with a dot product



and consider the operator



here





 (44)

in which the potentials have the form



with domain



 Then the problem can be interpreted as a study of the operator   on a non-compact graph. The spectral problem can be described as follows.

Find a vector  where   is a solution to the Dirac equation

                                (45)

on an edge  for which the following conditions are satisfied:

* 1.   is continuous at the nodes of the graph, in particular, in our case



* 1. The sum of derivatives over all branches outgoing from a node, calculated for each node is equal to zero



As a solution to the problem, we will understand the matrix



on a non-compact graph based on the following conditions.

1. 
2. 
3. 

where are the Jost solutions on the infinite edge of the graph and are determined by using the following theorem.

**Theorem 21**. Let  be defined according to (44). Then equation (44) has particular solutions that can be represented in the form of 





here   and   where the numbers   are determined using recurrence relations for which the series converges

 .

The inverse problem of recovering a differential operator on each edge is considered. Namely,

**Inverse problem**. Construct the potentials  and  given spectral data, the reflection coefficients  on each edge 

              **Theorem 22**. Specification of spectral data uniquely determines the potentials  and on each edge 

On each edge   the following relations are fulfilled



Finally using the formulas





we reconstruct the potentials  and  from the given spectral data, the reflection coefficients on each edge

**CONCLUSIONS**

  The dissertation work is devoted to the study of spectral problems for differential operators with complex-valued periodic potential. Inverse problems for various types of discontinuous differential operators with complex-valued periodic potential are studied. Scattering problems on quantum graphs are investigated.

The following main results were obtained in the dissertation.

- Necessary and sufficient conditions are obtained for a given sequence of complex numbers to be a set of spectral data for a second-order operator pencil and higher-order differential operators.

- The direct and inverse problem of spectral analysis for  order ordinary differential equations with a polynomially depending spectral parameter has been solved. It is shown that the spectrum of the operator pencil is continuous and fills the rays  where   and on the continuous spectrum there are spectral features that coincide with numbers of the form . The inverse problem of recovering the coefficients is solved by the generalized normalization numbers.

- Solved the inverse problem for the Schrödinger operator with complex-valued periodic potentials and a discontinuous coefficient on the whole real axis. The main characteristics of fundamental solutions are investigated, the spectrum of the operator is studied. The inverse problem is formulated, the uniqueness theorem is proved, and a constructive procedure for solving the inverse problem is proposed

- The classical Hill problem with complex potential is extended to star and loop graphs. The definition of the Hill operator on such graphs is given. The operator is defined by complex, periodic potentials and special boundary conditions are used to connect the values ​​of the functions at the vertices. An explicit description of the form of the resolvent is given, and the spectrum is described, and the inverse problem for reflection coefficients is solved.

**The main results of the dissertation were published in the following works:**

1. Эфендиев, Р.Ф. Существование операторов преобразования для дифференциальных уравнений, полиномиально зависящих от параметра//Труды конференции, посвященной 80-летию К.Т.Ахмедова, -Баку: -30-31 октября, -1997, -с.81-82.
2. **Efendiev, R.F.** To the spectral analysis of ordinary differential operators polynomially depending on a spectral parameter with periodic matrix coefficients. //-Baku: Proceedings of Institute of Mathematics and Mechanics, Academy of Science of Azerbaijan, -2000, v.12(20), p.30-34.
3. Эфендиев, Р.Ф. К спектральному анализу обыкновенных дифференциальных операторов, полиномиально зависящих от спектрального параметра с периодическими матричными коэффициентами//Theory and Practice of Differential Equations, Mathematical Research, -Saint-Petersburg: - 2000, с.142-146.
4. Эфендиев, Р.Ф. Спектральный анализ одного класса несамосопряженных дифференциальных операторов второго порядка//The Third International Conference “Tools for mathematical modelling”, -Saint-Petersburg, -18-23 june, -2001,-p.124.
5. **Эфендиев, Р.Ф.** Обратная задача для одного класса дифференциальных операторов второго порядка//-Баку: Доклады НАН Азерб., -2001, v.**57** № 4-6, -p.15-20.
6. Efendiev, R.F. Inverse problem for a class of second order differential operator//General Guide & Abstracts of Third Joint Seminar on Applied Mathematics, -Baku: Baku State Univers. & Zanjan Univers., -6-8 September -2002, -p.107.
7. Efendiev, R.F. Inverse problem for a class of ordinary differential operators with periodic coefficients // **Conference on ILL-POSED and INVERSE PROBLEMS** in honour of the 70-th anniversary of the birth of prof. **M.M.Lavrent'ev,-**Russia, Novosibirsk**: -**5-9 **A**ugust, -2002, -p.
8. Эфендиев, Р.Ф. Необходимые и достаточные условия решения обратной задачи для операторных пучков с комплексными периодическими коэффициентами// Международ-ная научная конференция «Современные проблемы математики и механики, информатики» посв., 80-летию проф. Л.А.Толокон-никова, -Россия, Тула: -18-21 ноября -2003, -с.49-51
9. Эфендиев, Р.Ф. Обратная задача для одного класса несамосопряженных опертаорных пучков с периодическими коэффициентами//»Функциональные пространства. Дифферен-циальные операторы. Проблемы математического образования», посв. 80-летию чл.-корр. РАН, проф. Л.Д.Кудрявцева, -Москва: -2003, -с.247-249
10. **Эфендиев, Р.Ф.** Обратная задача для одного класса операторов с комплексными периодическими коэффициентами//- Баку: Доклады НАН Азерб, -2004, №1-2, -p.39-43.
11. **Эфендиев, Р.Ф.** Обратная задача для одного класса обыкновенных дифференциальных операторов с периодичес-кими коэффициентами//-Kharkov: Математическая физика, анализ и геометрия, -2004, т.**11,** №1, -с.114-121 **(Impact factor 0.424)**.
12. **Эфендиев, Р.Ф.** Об одной задаче характеризации для пучка операторов с комплексными периодическими коэффициентами//-Баку: Доклады НАН Азерб, -2005, №3, -p.6-11.
13. **Эфендиев, Р.Ф. Спектральный анализ одного класса несамосопряженных дифференциальных операторных пучков с обобщенной функцией//-Москва: Теоретическая и математичес-кая физика, -2005, т.145, №1, -с. 102-107.**
14. **Èfendiev, R.F.** Spectral analysis of a class of nonselfadjoint differential operator pencils with a generalized function//-Moscow: Theoretical and Mathematical Physics, -2005, v.145, №1, -p.1457–1461**(Impact factor 0.901)**.
15. **Efendiev, R.F.** Complete solution of an inverse problem for one class of the high order ordinary differential operators with periodic coefficients//Journal of Mathematical Physics Analysis and Geometry, -2006, v.**2**, №1, -p.73-86. **(Impact factor 0.424)**.
16. Efendiev, R.F. Inverse wave scattering with discontinuous wave speed// International conference **«Inverse and III-Posed Problems of Mathematical Physics»**, dedicated to prof. M. M.Lavrent’ev on the occasion of his 75-th birthday, -Russia, Novosibirsk: -20-25 August, -2007, -1p.
17. **Efendiev, R.F.** The characterization problem for one class of second-order operator pencil with complex periodic coefficients//-Moscow: Moscow Mathematical Journal, -2007, v.7, №1, -p.55–65. **(Impact factor 0.746)**.
18. Efendiev, R.F, Inverse spectral problem for a differential operator with discontinuity wave speed//-Baku: Report of National Academy of Science Azerbaijan, -2008, №3, -p. 6-11.
19. Efendiev, R.F. An iterative algorithm for the solution of the discrete periodic optimal regulator problem/ F.A.Aliev, N.A.Safarova, A.Nachaoui, Y.S.Gasimov, R.F.Efendiev// -Baku: Report of National Academy of Science Azerbaijan**,** -2008, v. LXIV, № 6, –p.1-16.
20. **Эфендиев, Р.Ф.** Обратная задача для дифференциального оператора второго порядка с разрывными коэффициентами// -Баку: Вестник Бакинского Государственного Университета, -2008, №4, -p.17-22.
21. Efendiev, R.F. Spectral analysis of nonselfadjoint Hill operators with a step like Potentials //International conference on Functional Analysis dedicated to 90-th anniversary of V.E.Lyantse. - Lviv, Ukraine: -17-21 November, -2010, -p.37-38.
22. Efendiev**, R.F., Orudzhev, H.D.** Inverse wave spectral problem with discontinuous wave speed, //-Ukraine: Journal of Mathematical Physics, Analysis, Geometry, -2010, v.6**,** №3, -p.255- 265. **(Impact factor 0.424)**.
23. Orucov, H.D., Efendiev, R.F. Spectral analysis of nonselfadjoint Hill operators // -Baku: Journal of Qafqaz University, Mathem. and comp. sciences -2010, № 29, v.1, -p. 47-54.
24. Efendiev, R.F. Spectral analysis for one class of second-order indefinite nonselfadjoint differential operator pencil// Journal [Applicable Analysis](http://www.tandfonline.com/toc/gapa20/current), -2011, v. 90, №12, -p.1837-1849. **(Impact factor 1.076)**.
25. Orucov, H.D., Efendiev, R.F. Spectral analysis of nonselfadjoint hill operator with step-like potentials // -Baku: Journal of Qafqaz University, -2011, №31, -p. 8-15.
26. Efendiev, R.F. Wave propagation in a one-dimensional layered-inhomogeneous medium with a barrier// The 4-th Congress of the Turkic World Mathematics Society, -Baku, -1-3 July, -2011, -p.192
27. Orujov, H.D., Efendiev, R.F. Spectral analysis of the wave equation in a one-dimensional layered inhomogeneous medium with barrier//-Baku: Journal of Qafqaz University, -2012, № 33,-p. 45-49.
28. Niftiyev, A.A., Efendiev, R.F., Alisheva, K.I. Variable domain eigenvalue problems for the Laplace operator with density //-Baku: Journal of Contemporary Applied Mathematics, -ISSN: 2222-5498, -2012, v.2, №1, -p.
29. **Niftiyev, A.A.**, **Efendiev, R.F.** Variable domain eigenvalue problems for the Laplace operator with density// International Journal of Nonlinear Science, -2013, v.**16**, №3, -p.280-288.
30. Оруджев, Г.Д., Эфендиев, Р.Ф. Спектральный анализ одного несамосопряженного операторного пучка с разрывными коэффи-циентами//-Киев: Доклады НАН Украины, -2014, №4, -с. 25-32.
31. Efendiev R.F, Garcia-Raffi, Luis M. Spectral analysis of the complex hill operator on the star graph//-Baku: Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, -2014, v.40, special issue, -p.124–132.
32. Orudzhev, H.D., Efendiev, R.F. Recovering of the Hill operator by two spectra //-Baku: Journal of Qafqaz University, -2015, v.3, № 2, -p. 8-15.
33. Efendiev, R.F., Orudzhev, H.D., Zaki, FA El-Raheem. Spectral analysis of wave propagation on branching strings//-London: Boundary Value Problems, -2016, 215, -p.1-18 DOI: 10.1186/s13661-016-0723-3. **(Impact factor 1.637)**.
34. Efendiev, R.F. Spectral analysis of Hill operator on lasso shaped graph. // International conference on “Operators in Morrey-type spaces and applications” dedicated to 60-th birthday of prof. V.S.Guliyev, -Kirsehir-Turkey: 10-13, July -2017, -p.153. (OMTSA-2017).
35. Efendiev, R.F. Inverse spectral problem for hill operator on lasso graph//The 6-th international conference on control and optimization with Industrial Applications, -Baku: 11-13 July, -2018, -p.150-151.

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