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APPROXIMATION BY A SUM OF SUBALGEBRAS OF THE SPACE OF CONTINUOUS FUNCTIONS

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Applicant: **Aida Khurshud kizi Asgarova**

ABSTRACT

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Scientific supervisor: dr. math. sci., assoc. prof

Vugar Elman oglu Ismailov

Official opponents: dr. math. sci., assoc. prof.

Rovshan Alifaga oglu Bandaliyev

dr. of ph. and math. sci., prof.

Alik Malik oglu Najafov

cand. of ph. and math. sci., assoc. prof.

Fuad Agca oglu Abdullayev

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan

Chairman of the Dissertation council: Corr. Member of ANAS,

dr. of ph. and math. sci., prof.

Misir Jumail oglu Mardanov

Scientific secretary of the Dissertation council:

cand. of ph. and math. sci.

_________ **Abdurrahim Farman oglu Guliyev**

Chairman of the scientific seminar: Corr. member of ANAS,

dr. of ph. and math. sci., prof.

_________ **Bilal Telman oğlu Bilalov**

GENERAL DESCRIPTION OF WORK

Relevance and degree of development of the topic. In 1885, Carl Weierstrass showed that every continuous function on $[a,b]$ can be approximated with any accuracy by the simplest functions, namely polynomials. Marshall Stone generalized this theorem by proving that any continuous function defined on a compact Hausdorff space X can be approximated by functions from some algebra of continuous functions if and only if this algebra separates points of the space X and does not disappear (does not turn into 0) at any point. This theorem generalizes the classical Weierstrass theorem in two directions: instead of the interval $[a,b]$, we take any Hausdorff compact X, and instead of the class of polynomials we consider a more general class of functions from the space of continuous functions. This important theorem is known in the literature as the Stone-Weierstrass theorem. Note that above and in the dissertation, we are talking about functions that take real values.

At present, many interesting results generalizing and strengthening the classical Stone-Weierstrass theorem (for a series of such results, see, for example, the works of E. Bishop, S. Machado, S. Boel, T. Carlsen, N. Hansen, P. Chernoff, R. Rasala, W. Waterhouse, J. Prolla, K. Srikanth, R. Yadav, R. Stephenson). Among the works of Azerbaijani mathematicians devoted to generalizations and analogs of the Stone-Weierstrass theorem, the works of B. Bilalov, V. Mirzoyev, and A. Turovsky should be noted.

One important direction generalizing Stone's ideas is the transition from a single algebra of continuous functions to the sum of several algebras. Let X, as above, be a compact Hausdorff space, $C(X)$ the space of real continuous functions on X and $A_i, i=1,...,k$, some subalgebras of the space $C(X)$.

Consider the approximation of functions from $C(X)$ by functions from the subspace $A_1+\cdots+A_k$. Concerning this approximation, the following natural problems arise:

1) Under what conditions does the sum $A_1+\cdots+A_k$ coincide with the whole space $C(X)$?

2) Under what conditions is the sum $A_1+\cdots+A_k$ everywhere dense in the space $C(X)$?

3) How to calculate the approximation error of a specific function $f\in C(X)$ by functions from $A_1+\cdots+A_k$?

4) How to characterize the best approximation (extremal element) from $A_1+\cdots+A_k$ to a given function $f\in C(X)$?

For problems (1) and (2) both sufficient and necessary conditions are interesting. It should be noted that the list of the above problems can be expanded. From the point of view of approximation theory are interesting, for example, the closed and proximinal problems of the sum $A_1+\cdots+A_k$ in $C(X)$, the problem of describing the annihilator $(A_1+\cdots+A_k)^{\perp}$, i.e. function spaces annihilating each element of $A_1+\cdots+A_k$, etc.

It should be noted that the circle of problems regarding the sum of subalgebras from $C(X)$ arose after the famous superposition theorem of A.N. Kolmogorov. This theorem in terms of algebras reads as follows. For a single cube $\mathbb{I}^d, \mathbb{I}=[0,1], d\geq 2,$ there are $2d+1$ algebras $A_i\subset C(X),$ $i=1,...,2d+1,$ such that that $C(\mathbb{I}^d)=$ $A_1+\cdots+A_{2d+1}$. Recall that each algebra in this theorem is generated by one eigenelement, i.e. for A_i there exists an element $s_i\in A_i$ such that $A_i=\{g_i\circ s_i : g_i\in C(\mathbb{R})\}$. This theorem began the study of many problems of approximation by linear superpositions and sums of subalgebras of the space of continuous functions. After the Kolmogorov theorem, many important and interesting works appeared, among which the works of Lorentz, Friedman, Vitushkin and Henkin, Ostrand, Sternfeld, Sprecher, Sproston and Straus, Marshall and O’Farrell, Khavinson, Ismailov should be noted. Basically, these works were aimed at generalizing, strengthening and obtaining analogues of Kolmogorov’s superposition theorem and at studying the properties of sums of arbitrary algebras of continuous functions. For example, Ostrand generalized Kolmogorov’s theorem to compact metric spaces. He proved that for every d -dimensional compact metric space X there exist continuous functions $\{\alpha_i\}_{i=1}^{2d+1} \subset C(X)$ such that $C(X) = A_1 + \cdots + A_{2d+1}$, where the algebras $A_i = \{g_i \circ \alpha_i : g_i \in C(\mathrm{R})\}$, $i = 1,...,2d+1$. Sternfeld showed that the number $2d+1$ cannot be reduced for any d -dimensional metric space X. Moreover, he gave a characteristic of the dimension of a metric compact in terms of sums of algebras. He showed that a compact metric space X has dimension d if and only if there exist $2d+1$ subalgebras of $A_i \subset C(X)$ such that $C(X) = A_1 + \cdots + A_{2d+1}$ and for all less than $2d+1$ subalgebras $B_i \subset C(X)$, $i = 1,...,k$, $k < 2d+1$, there is $C(X) \neq B_1 + \cdots + B_k$. From this result it follows that the number of terms in Kolmogorov’s superposition theorem is the best.

In the works of Sternfeld, the study of the properties of the sums of subalgebras of the space of continuous functions was begun as an independent direction. Kolmogov’s theorem and its generalizations state that for a particular topological space X (for a d -dimensional cube, d -dimensional compact set of Euclidean space, metric compact set of dimension d) there are algebras A_i, $i = 1,...,2d+1$, such that $C(X) = A_1 + \cdots + A_{2d+1}$. We can set a more general problem: Let X be a compact topological space and A_i, $i = 1,...,k$, of the subalgebra $C(X)$. Under what conditions $C(X) = A_1 + \cdots + A_k$? In this problem, conditions must be imposed on the family of algebras A_i. For example, for compact metric and compact Hausdorff spaces X Sternfeld and Sproston, Straus, respectively, have obtained a sufficient condition for the sum $A_1 + \cdots + A_k$ to coincide with the whole space $C(X)$, which is also necessary when $A_2 = \cdots = A_k$.

The dissertation continues the study of the cycle of problems around the general problem of approximation by the sums of subalgebras of the space of continuous functions. More precisely, the above problems (1) - (4) are studied and several theorems are proved that strengthen, complement and generalize the corresponding known results.

Object and subject of research. Spaces of continuous functions, the sums of the algebras of the space continuous functions.

The purpose and objectives of the research. 1) Investigate the problem of representing the space of continuous functions defined on a compact Hausdorff space as a sum of subalgebras of this space. To study the question of the density of such sums in the space of continuous functions.

2) Investigate the Diliberto-Straus algorithm for convergence in the problem of approximating a continuous function defined on a compact Hausdorff space by elements of the sum of two subalgebras of the space of continuous functions.

3) Find the necessary and sufficient condition for the element of the sum of algebras to be the best approximation (extremal element) for a given continuous function.

4) Apply the obtained general results to the study of approximating properties of concrete, important from the point of view of applications, algebras of functions.

Research methodology. The methods of the theory of functions, functional analysis, topology and approximation theory are applied in the work.

The main provisions submitted for defense.

1) Results on the representation of the space of continuous functions defined on a compact Hausdorff space as the sum of subalgebras of this space.

2) Results on the density of the sum of algebras in the space of continuous functions.

3) Result on the convergence of the Diliberto-Straus algorithm to the approximation error in the problem of approximating a continuous function by the sum of two subalgebras of the space of continuous functions on a compact Hausdorff space.
4) A result on the Chebyshev characterization of the best approximation by sums of algebras.
5) Applied results on the approximating properties of the sum of ridge function algebras. A formula for calculating the error of approximation of a function of many variables by elements of the sum of two ridge algebras.

The scientific novelty of the research. 1) The necessary conditions are found for representing the space of continuous functions defined on a compact Hausdorff space as the sum of the subalgebras of this space. These conditions supplement the sufficient condition of Sproston and Straus for the possibility of such a representation.

2) It is proved that for the case of two algebras, the above necessary condition is also sufficient. The obtained result generalizes the classical Stone-Weierstrass theorem from the case of one algebra to the case of the sum of two algebras.

3) A necessary condition is found for the density of the sum of algebras in the space of continuous functions, which, under the corresponding restrictions, is also sufficient.

4) The convergence of the Diliberto-Straus algorithm to the approximation error in the problem of approximating a continuous function by the sum of two subalgebras of the space of continuous functions on a compact Hausdorff space is proved.

5) A necessary and sufficient condition is found for the element of the sum of algebras to be the best approximation (extreme element) for a given continuous function. The result obtained is an analog of the Chebyshev theorem on characterizing a polynomial of best approximation.

6) The general results obtained are applied to the study of the approximating properties of the sum of ridge function algebras. A formula is established for calculating the error of approximation of a

function of many variables by elements of the sum of two ridge algebras.

The theoretical and practical significance of the research. The work is theoretical. The results of the work can be used in function theory, functional analysis, approximation theory, and in areas where the approximation of complex functional dependencies is required through simple and at the same time useful functions, such as ridge functions, radial functions, etc.

Approbation and implementation. The main results of the dissertation were reported at the Institute of Mathematics and Mechanics of the NAS Of Azerbaijan at the seminars of the department "Functions theory" (head, prof. V.E. Ismailov), as well as at national and international scientific conferences "Actual problems of Mathematics and Mechanics" dedicated to the 94th birthday of national leader Heydar Aliyev, "Modern problems of mathematics and mechanics" international conference devoted to the 80th anniversary of academician Akif Gadjiev (6-8 december, Baku 2017), “Differential equations and related problems”, international conference dedicated to the anniversaries (70th anniversary) of Academician of the Russian Academy of Sciences Moiseev E. I., Academician of the Academy of Sciences of the Republic of Belarus Shagapova V. Sh. and Professor Soldatov A. P. (25-29 June, Sterlitamak, Ufa 2018), “Modern Problems of Innovative Technologies and Applied Mathematics in Oil and Gas Production” International Conference devoted to the 90th anniversary of Azad Mirzajanzade (13-14 december, Baku 2018), “Complex Analysis and Approximation Theory” (29 – 31 May, Ufa 2019), International Conference “Modern Problems of Mathematics and Mechanics devoted to the 60th anniversary of the Institute of Mathematics and Mechanics” (23-25 October, Baku 2019).

Personal contribution of the author. All proved theorems in the dissertation, namely theorem on the representation space continuous functions defined on compact Hausdorff space as the sum of subalgebras of this space; theorem on the density of the sum of algebras in the space of continuous functions theorem on the convergence of algorithm Diliberto-Strauss to the error of the approximation in the problem of approximation of continuous functions elements of the sum of two subalgebras of the space of continuous functions; the theorem on the characterization chebyshevskii best approximation of sums of algebras; the theorem approximating properties of the sum of the ridge function algebras; a theorem on calculating the error of the approximation of a function of many variables by elements of the sum of two ridge algebras obtained by the author.

The name of the organization where the dissertation was performed. Institute of Mathematics and Mechanics Azerbaijan National Academy of Science.

Publications. The results of the dissertation are published in 15 papers (8 articles, 7 of them are indexed in the Web of Science platform of Clarivate Analytics (3 in the "Science Citation" database Index Expanded", 4 in the database "Emerging Sources Citation Index") and 7 abstracts (6 of them were published at international scientific conferences and 1 at the Republican scientific conference)), a list of which is given at the end of the abstract.

Structure and scope of work. The dissertation consists of an introduction (~ 48000), 3 chapters (Chapter I ~ 52000, Chapter II ~ 76000, Chapter III ~ 44000), and a list of references including 116 names. The total volume of the dissertation is 125 pages (~ 220000 signs).

CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, three chapters, and a list of references.

Chapter 1 discusses the problems of representing the space of continuous functions as a sum of its subalgebras and as closure of such a sum. This chapter consists of five sections. Section 1.1 describes the statement of the problem of representation by sums of algebras and discusses some well-known necessary or sufficient conditions for the possibility of such a representation. Let X be a compact Hausdorff space and $C(X)$ the space of real continuous functions on X, equipped with the topology of uniform convergence. Suppose also that we are given a finite number of closed subalgebras $A_1,...,A_k$ of the space $C(X)$ that contain constant functions. Consider the following problem. What conditions imposed on $A_1,...,A_k$ are necessary and / or sufficient for the equality $C(X)=A_1+\cdots+A_k$?

In this section, first, two simple necessary conditions for this equality are considered alternately, exactly, the following conditions are considered:

a) for any different $x,y\in X$ there exists an algebra A_i separating these points;

b) for any closed sets $P,Q\subset X$, $P\cap Q=\varnothing$, there exists an algebra A_i separating these sets.

It is shown in turn that these conditions are not sufficient for the representation $C(X)=A_1+\cdots+A_k$. Then, one sufficient condition of Sproston and Straus is given, which is necessary only for the case $k=2$.

Section 1.2 defines new objects called the *cycle* and *half-cycle* with respect to a finite family of subalgebras of the space of continuous functions, as well as the functionals generated by these objects. In order to determine the corresponding objects, first, we consider the equivalent relations R_i, $i=1,...,k$, for points in the space X:

$$a\overset{R_i}{\sim}b \text{ if } f(a)=f(b) \text{ for all } f\in A_i$$

Since the space X itself is compact, for each $i=1,...,k$, the factor is the space $X_i=X/R_i$ (relative to the relation R_i) equipped with factor topology is compact. In addition, the canonical projections $s_i:X\to X_i$ are continuous mappings.

It follows from the Stone-Weierstrass theorem that each of the algebras A_i, $i=1,...,k$, (as a set) can be represented as

$$A_i=\{f(s_i(x)): f\in C(X_i)\}, i=1,...,k.$$

Cycles and *semicycles* with respect to the algebras A_i, $i=1,...,k$, are defined as follows.

Definition 1. *A set of points* $l=(x_1,\dots,x_n)\subset X$ *is called a cycle with respect to the algebras* A_i, $i=1,...,k$, *if there exists a vector* $\lambda=(\lambda_1,\dots,\lambda_n)\in \mathbb{Z}^n$ *with the nonzero integer coordinates* λ_j *such that*

$$\sum_{j=1}^{n}\lambda_j\delta_{s_i(x_j)}=0, \quad for\, all\, i=1,\dots,k.$$

Here δ_a is a characteristic function of the unit set $\{a\}$.

For example, the set $l=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$ is a cycle in $\mathbb{I}^3$, $\mathbb{I}=[0,1]$, with respect to the algebras $A_i=\{p(z_i): \ p\in C[0,1]\}, i=1,2,3.$ The vector λ in Definition 2.1 can be taken as $(-2,1,1,1,-1)$.

Definition 2. *A set of points* $l=(x_1,\dots,x_n)\subset X$ *is called a semicycle with respect to the algebras* A_i, $i=1,...,k$, *if there exists a vector* $\lambda=(\lambda_1,\dots,\lambda_n)\in \mathbb{Z}^n$ *with the nonzero integer coordinates* λ_j *such that for any* $i=1,...,k$,

$$\sum_{j=1}^{n}\lambda_j\delta_{s_i(x_j)}=\sum_{t=1}^{r_i}\lambda_{i_t}\delta_{s_i(x_{i_t})}, \quad where\, r_i\le k.$$

Note that for $i=1,\dots,k$, the set $\{\lambda_{i_t}, t=1,...,r_i\}$ is a subset of the set $\{\lambda_j,\ j=1,...,n\}$. This means that for each i we have at most k terms in the sum $\sum_{j=1}^{n}\lambda_j\delta_{s_i(x_j)}$ (the sum of the remaining terms identically vanishes). We give one example of a semicycle. We are given two algebras A_1 and A_2 with quotient mappings s_1 and s_2 respectively. Let $l=\{x_1,x_2,...,x_n\}$ be a set with the following property:

$$s_1(x_1) = s_1(x_2), s_2(x_2) = s_2(x_3), s_1(x_3) = s_1(x_4), ..., s_2(x_{n-1}) = s_2(x_n).$$

Then the set $l = \{x_1, \ldots, x_n\}$ is a semicycle with respect to the algebras A_1 and A_2.

Note that in Marshall and O'Farrell, a finite sequence $(x_1,..., x_n)$ with $x_i \neq x_{i+1}$ satisfying either $s_1(x_1) = s_1(x_2)$, $s_2(x_2) = s_2(x_3)$, $s_1(x_3) = s_1(x_4),...,$ or $s_2(x_1) = s_2(x_2)$, $s_1(x_2) = s_1(x_3)$, $s_2(x_3) = s_2(x_4),...,$ is called a *bolt* with respect to (A_1, A_2). Similarly, infinite bolt $(x_1, x_2,...)$ is defined. If $(x_1,..., x_n, x_1)$ is a bolt and n is an even number then the bolt $(x_1,..., x_n)$ is called closed.

In Section 1.2 introduces one more definition.

Definition 3. *A cycle (or semicycle) l is called a q-cycle (q-semicycle) if the vector λ associated with l can be chosen so that $|\lambda_i| \leq q$, $i = 1,..., n$, and q is the minimal number with this property.*

The semicycle considered above is a 1-semicycle. If in that example, $s_2(x_{n-1}) = s_2(x_1)$, then the set $\{x_1, x_2,..., x_{n-1}\}$ is a 1-cycle. Let us give a simple example of a 2-cycle with respect to the algebras $U = \{u(x)\}$, $V = \{v(y)\}$ considered above. Consider the union of the vertices of two squares $[0,1]^2$ and $[0,2]^2$, that is, a set of:

$$\{(0,0),(1,1),(2,2),(0,1),(1,0),(0,2),(2,0)\}.$$

Clearly, this set is a 2-cycle with the associated vector $(2,1,1,-1,-1,-1,-1)$. Similarly, one can construct a q-cycle or q-semicycle for any positive integer q.

Each semicycle $l = (x_1, \ldots, x_n)$ and an associated vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ generate the following functional

$$F_{l,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad f \in C(X).$$

Obviously, $F_{l,\lambda}$ is a bounded linear functional with the norm $\sum_{j=1}^{n}|\lambda_j|$.

In Section 1.3 establishes the necessary condition for representing the space of continuous functions defined on a compact Hausdorff space by the sums of its k closed subalgebras. A sufficient condition of this kind for the presentation was obtained by Y.Sternfeld in 1978. In this section, at first, the following lemma is proved:

Lemma 1. *If the semicycle* $l=\{x_1,\dots,x_n\}$ *is a cycle, then* $F_{l,\lambda}(g)=0$*, for each element* $g\in A_1+\cdots+A_k$.

Then the main result is proved:

Theorem 1. *Let* $A_1+\cdots+A_k=C(X)$. *Then*

(Z_1) *there are no cycles in* X *;*

(Z_2) *for each* $q\in\mathbb{N}$*, the lengths (number of points) of all* q*-semicycles in* X *are uniformly bounded.*

In Section 1.4 proves that for the sum of two subalgebras ($k=2$), the conditions (Z_1) and (Z_2) are also sufficient. In the case of one subalgebra ($k=1$), the result obtained coincides with the classical Stone-Weierstrass theorem.

Theorem 2. *If* $k=2$, *then the conditions* (Z_1) *and* (Z_2) *together are both necessary and sufficient for the representation* $A_1+\cdots+A_k=C(X)$. *Moreover, in* (Z_2)*, the consideration of only* 1-*semicycles suffices.*

Theorem 2 can be formulated in the following form, which is more convenient for applications.

Theorem 3. *The representation* $C(X)=A_1+A_2$ *holds if and only if the space* X *with respect to* (A_1,A_2) *and length of all bolts are uniformly bounded.*

For the case of a compact set $X\subset\mathbb{R}^2$ and the algebras $U=\{u(x)\}$, $V=\{v(y)\}$ of univariate functions defined on the projections of X into the coordinate axes x and y, respectively, Theorem 3 was first obtained by Khavinson. Using the results of Sternfeld, Khavinson in his monograph proved this result also for the case of linear superpositions.

Further in this section, it is proved that the Stone-Weierstrass theorem is a consequence of Theorem 3 and the conditions of Theorem 1 are not sufficient to represent $C(X) = A_1 + \cdots + A_k$ if $k > 2$.

Section 1.5 gives one necessary condition for representing the space of continuous functions as the closure of the sum of a finite number of closed algebras. Under appropriate restrictions, the results obtained positively solve the question of the possibility of approximating a continuous function by elements of the sum of algebras with arbitrary accuracy. The main result of this section is as follows.

Theorem 4. *Let* $\overline{A_1 + \cdots + A_k} = C(X)$. *Then in the space* X *there exist no cycles with respect to the family of algebras* A_i, $i = 1,...,k$.

In **Chapter 2** the problem of best approximation of a continuous function defined on a compact Hausdorff space by the sum of two closed algebras is considered and the convergence of the Diliberto-Straus algorithm to the approximation error is examined. In addition, a theorem of Chebyshev type is proved for characterizing the best approximation (extremal element). This chapter consists of five paragraphs. In Section 2.1 the statement of the problem is presented and some well-known results on the convergence of the Diliberto-Straus algorithm for various approximation problems are provided. Let X be a compact Hausdorff space, $C(X)$ the space of real continuous functions on X and $A_1 \subset C(X)$, $A_2 \subset C(X)$ closed algebras that contain constant functions. Consider the approximation of the function $h \in C(X)$ by the elements of the sum $A_1 + A_2$ (or simply the sum $A_1 + A_2$). We construct the following functional sequence:

$$h_1(x)=h(x), h_{2n}=h_{2n-1}-Fh_{2n-1}, h_{2n+1}=h_{2n}-Gh_{2n}, n=1,2,...,$$

where $F:C(X)\to A_1$ and $G:C(X)\to A_2$ are best approximation operators. The process of releasing members of this sequence is called the Diliberto-Straus algorithm. It is clear that $\|h_1\|\geq \|h_2\|\geq\|h_3\|\geq\cdots\geq E(h)$, where $E(h)$ is an approximation error, i.e.

$$E(h)=\inf_{w\in A_1+A_2}\|h-w\|.$$

The problem is as follows: under what conditions does the Diliberto-Straus algorithm converge to an approximation error, i.e. $\|h_n\|\to E(h)$, for $n\to\infty$?

To solve this problem, in Section 2.2 we study the question of the continuity of the maximum and minimum functions that are generated using canonical mappings $s:X\to X_1$, $p:X\to X_2$, where X_1 and X_2 the factor space generated by the equivalent relations

$a\overset{R_i}{\sim}b$ if $f(a)=f(b)$ forall $f\in A_i$, $i=1,2$, respectively.

Let $h\in C(X)$. Consider real functions

$$f_1(a)=\max_{\substack{x\in X\\ s(x)=a}}h(x),\ f_2(a)=\min_{\substack{x\in X\\ s(x)=a}}h(x), a\in X_1,$$

$$g_1(b)=\max_{\substack{x\in X\\ p(x)=b}}h(x), g_2(b)=\min_{\substack{x\in X\\ p(x)=b}}h(x), b\in X_2.$$

When are the functions f_i and g_i continuous on the sets X_1 and X_2, respectively? The continuity of these functions strictly depends on the map factor s and p. If the functions f_i and g_i are continuous for each $h\in C(X)$, then we say that the algebras A_1 and A_2 have C -property. The following theorem holds with respect to the C -property.

Theorem 5. *Let X be a compact sequential Hausdorff space and A be a closed subalgebra of $C(X)$ that contains the constants. Let X_1 be a quotient space generated by the equivalence relation* *and* $s: X \to X_1$ *be the natural quotient mapping. Then the functions* f_1 *and* f_2 *are continuous on* X_1 *for any* $h \in C(X)$ *if for any two points* x *and* y *with* $s(x) = s(y)$ *and any sequence* $\{x_n\}_{n=1}^{\infty}$ *tending to* x*, there exists a sequence* $\{y_n\}_{n=1}^{\infty}$ *tending to* y *such that* $s(y_n) = s(x_n)$*, for all* $n = 1,2,..$

Further, a corollary of this theorem is proved for a concrete algebra important from the point of view of applications ridge functions:

Corollary 1. *Let* Q *be a compact convex set in* $\mathbb{R}^d$*, then the algebra* $A = \{g(\mathbf{a}\cdot\mathbf{x})\}$*, where* $\mathbf{a}$ *nonzero fixed vector in* $\mathbb{R}^d$ *and functions of the form* $g(\mathbf{a}\cdot\mathbf{x})$ *(called a ridge functions) vary in* $C(Q)$*, holds* C*-property.*

From this corollary, in a particular case, it follows that the algebras $A_1 = \{f(x): \ f \in Q_x\}$ and $A_2 = \{g(y): \ g \in Q_y\}$, where Q_x and Q_y are projections of compact set Q $Q \subset \mathbb{R}^2$ into the coordinate axes x and y, respectively holds C-property.

Section 2.3 the operators are defined

$$F: C(X) \to A_1,\ Fh(a) = \frac{1}{2}\left(\max_{\substack{x\in X \\ s(x)=a}} h(x) + \min_{\substack{x\in X \\ s(x)=a}} h(x) \right), \text{all}\, a \in X_1,$$

$$G: C(X) \to A_2,\ Gh(b) = \frac{1}{2}\left(\max_{\substack{x\in X \\ p(x)=b}} h(x) + \min_{\substack{x\in X \\ p(x)=b}} h(x) \right), \text{all}\, b \in X_2.$$

and the theorem is proved:

Theorem 6. *Assume* X *is a compact Hausdorff space and* A_i*,* $i = 1,2$*, are closed subalgebras of* $C(X)$*, that contain the constants. In addition, assume that the* C*-property holds for these subalgebras. Then the operators* F *and* G *are central operators of best approximation acting from the space* $C(X)$ *onto* A_1 *and* A_2*, respectively. In addition, these operators are non-expansive, i.e. the* *following inequalities are satisfied for them:*

$$\|Fv_1 - Fv_2\| \le \|v_1 - v_2\| \text{ and } \|Gv_1 - Gv_2\| \le \|v_1 - v_2\|$$

for all $v_1, v_2 \in C(X)$.

From this theorem and the general result of Golomb we obtain the convergence of the Diliberto-Straus algorithm for approximation by the sum of two algebras under certain conditions, as shown by the following theorem.

Theorem 7. *Let* X *be a compact Hausdorff space and* A_i, $i = 1,2$, *are closed subalgebras of* $C(X)$, *that contain the constants. Assume, in addition, that the* C*-property holds for these subalgebras.* $A_1 + A_2$ *be closed in* $C(X)$. *Then* $\|h_n\|$ *converges to the error of approximation* $E(h) \overset{def}{=} dist(h, A_1 + A_2)$.

Section 2.4 gives a new proof of Theorem 7 by applying completely new ideas. In this proof you can see why you have to resort to the closure of the sum of algebras. It should also be noted that in the classical method for proving the convergence of an algorithm for approximating by sums of functions of one variable, each lightning $(x_1,..., x_n)$ of the compact set $Q \subset \mathbb{R}^2$ must be closed (that is, to form a closed lightning) by adding one point $y \in Q$, whose coordinates are equal to the first coordinate of the point x_1 and the second coordinate of the point x_n, respectively. Obviously, this point may not lie in the set Q, if Q is not the Cartesian product of two segments. That is why the classical result of Diliberto and Straus and some of his generalizations featured the Cartesian product of two defined sets. The main idea of our proof is to apply the special property of bolt, which cannot be closed by adding a point. Namely, we introduce $*$ -weak limit points of a sequence of functionals generated by an unclosed bolt.

It follows from Theorem 7 and Corollary 1 that the classical result of Diliberto and Straus is valid not only for a rectangle with sides parallel to the coordinate axes, but also for a series of convex compact sets in the space $\mathbb{R}^2$. More precisely, the following corollary is true.

Corollary 2. *Let* $Q \subset \mathbb{R}^2$ *be a convex compact set,* $A_1 = \{f(x)\}$ *and* $A_2 = \{g(y)\}$ *be the algebras of univariate functions, which are continuous on the projections of* Q *into the coordinate axes* x *and* y*, respectively. Assume that any bolt can be made closed by adding only a fixed number of points of* Q*. Then for the function* $h \in C(Q)$*, the sequence*

$$\|h_1\| = \|h\|, \|h_{2n}\| = \|h_{2n-1} - Fh_{2n-1}\|, \|h_{2n+1}\| = \|h_{2n} - Gh_{2n}\|,$$

$n = 1,2,...,$ *converges in norm to the error of approximation from* $A_1 + A_2$.

In Section 2.5, a Chebyshev-type theorem for characterizing the best approximation in the problem of approximating a continuous function defined on some compact metrizable space by elements of the sum of two algebras is proved. Let's move on to the exact formulation of this problem. Let X be a compact metrizable space, $C(X)$ the space of real continuous functions on X and $A_1 \subset C(X)$, $A_2 \subset C(X)$ closed algebras that contain constant functions. With respect to the algebras A_1 and A_2, it is also assumed that these algebras hold C-property. Consider the approximation of the function $h \in C(X)$ by the elements of the sum $A_1 + A_2$ (or simply the sum $A_1 + A_2$). If there exists an element $w_0 \in A_1 + A_2$ such that

$$\inf_{w \in A_1 + A_2} \|h - w\| = \|h - w_0\|,$$

then this element is called the best approximation (or extremal element) for h. How can you characterize the best approximation? In other words, what conditions imposed on the function $w_0 \in A_1 + A_2$ are necessary and sufficient for this function to be the best approximation for a given function h? To solve this problem, first, a definition is introduced

Definition 4. *A finite or infinite path* $(x_1, x_2, ...)$ *with respect* *to* (A_1, A_2) *is said to be extremal for a function* $f \in C(Q)$, *if* $f(x_i) = (-1)^i \|f\|, i = 1,2,...$ *or* $f(x_i) = (-1)^{i+1}\|f\|$, $i = 1,2,...$.

For ridge functions and linear superpositions above definition was considered by Ismailov.

The main result of paragraph 2.5 is as follows:

Theorem 8. *Let* X *be a compact metrizable space and* $u \in C(X)$. *The function* $w_0 \in A_1 + A_2$ *is the best approximation for* f *if and only if there is a closed or infinite lightning with respect to* (A_1, A_2), *extremal for the function* $u - w_0$.

Chapter 3 considers concrete and important in terms of applications function algebras, namely ridge function algebras. Ridge functions have numerous applications in various fields. The general results obtained in previous chapters are valid for the sum of ridge function algebras. This chapter consists of three sections. Section 3.1 formulates some results for the case of ridge algebras, which are obtained from the theorems of the previous chapters. Section 3.2 proves a new theorem on finding the error of approximation by the sum of two ridge algebras.

Let Q compact set in $\mathbb{R}^d$, $d \geq 2$, and f any continuous function on Q. Consider approximation of function f with a sum of two ridge functions

$$\mathcal{A}_1 = \{g_1(\mathbf{a} \cdot \mathbf{x}) : g_1 \in C(\mathbb{R})\}$$

and

$$\mathcal{A}_2 = \{g_2(\mathbf{b} \cdot \mathbf{x}) : g_2 \in C(\mathbb{R})\}$$

where $\mathbf{a}$ and $\mathbf{b}$ are fixed directions in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\mathbf{x} \in Q$. For brevity, the sum $\mathcal{A}_1 + \mathcal{A}_2$ is denoted by $\mathcal{R}$.

Path with respect to the directions $\mathbf{a}$ and $\mathbf{b}$ is defined as an ordered set of points $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m)$ in $\mathbb{R}^d$ such that $\mathbf{x}_i \neq \mathbf{x}_{i+1}$ and the segments $\mathbf{x}_{i+1} - \mathbf{x}_i$ are alternately perpendicular to the directions $\mathbf{a}$ and $\mathbf{b}$. A finite path $(\mathbf{x}_1, ..., \mathbf{x}_m)$ is said to be closed if m is even number and the set $(\mathbf{x}_1,...,\mathbf{x}_m,\mathbf{x}_1)$ forms path in the indicated order. The path $(\mathbf{x}_1,...,\mathbf{x}_m)\subset Q$ is called reducible, if there exist points $\mathbf{y},\mathbf{z}\in Q$ such that $(\mathbf{y},\mathbf{x}_1,...,\mathbf{x}_m,\mathbf{z})$ is also a path.

With each closed path $p=(\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_{2n})$, we associate the functional

$$G_p(f)=\frac{1}{2n}\sum_{k=1}^{2n}(-1)^{k+1}f(\mathbf{p}_k).$$

This functional has the following obvious properties:

(a) If $g\in\mathcal{R}$, then $G_p(g)=0$.

(b) $\|G_p\|\le 1$. If $p_i\neq p_j$ for all $i\neq j$, $1\le i,j\le 2n$, then $\|G_p\|=1$.

The main result of this section is the following theorem.

Theorem 9. *Let $Q\subset\mathbb{R}^d$ be a convex compact set, $\mathbf{a},\mathbf{b}\in\mathbb{R}^d\setminus\{\mathbf{0}\}$ and $f\in C(Q)$. Assume the following conditions hold:*

1) there exists a best approximation $g_0\in\mathcal{R}$ for function f ;

2) for any expandable path $q=(\mathbf{q}_1,...,\mathbf{q}_n)\subset Q$ with respect to $\mathbf{a}$ and $\mathbf{b}$, there exist points $\mathbf{q}_{n+1},\mathbf{q}_{n+2},...,\mathbf{q}_{n+s}\in Q$ such that $(\mathbf{q}_1,...,\mathbf{q}_n,\mathbf{q}_{n+1},...,\mathbf{q}_{n+s})$ is a closed path and s is not more than some positive integer n_0, independent of q.

Then, for the approximation error of the function f by sums from $\mathcal{R}$, the following formula

$$E(f)=\sup_{p\subset Q}|G_p(f)|,$$

is valid, where the sup is taken over all closed paths with respect to directions $\mathbf{a}$ and $\mathbf{b}$.

It should be noted that if the space $\mathcal{R}$ is proximinal in $C(Q)$ (i.e., if for any continuous function on Q there exists a best approximation from $\mathcal{R}$), then the second condition of Theorem 9 is satisfied automatically. In other words, the following theorem holds.

Theorem 10. *Let $Q \subset \mathbb{R}^d$ be a convex compact set, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $f \in C(Q)$ and the subspace $\mathcal{R}$ is proximinal in $C(Q)$. Then, for the approximation error of the function f by sums from $\mathcal{R}$, the following formula*

$$E(f, \mathcal{R}) = \sup_{p \subset Q} |G_p(f)|,$$

is valid, where the $\sup$ is taken over all closed paths with respect to directions $\mathbf{a}$ and $\mathbf{b}$.

Section 3.3 gives a result similar to Theorem 9 for approximation by a sum of algebras of radial basis functions. The radial basis function is a function of the form

$$F(\mathbf{x}) = r(\|\mathbf{x} - \mathbf{c}\|),$$

where $r : \mathbb{R} \to \mathbb{R}$ univariate function, $\mathbf{x} = (x_1, ..., x_d)$ is the variable, $\mathbf{c} \in \mathbb{R}^d$ any fixed point, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. The point $\mathbf{c}$ is called the center of F. In other words, a radial basis function– is a multivariate function constant on the spheres $\|\mathbf{x} - \mathbf{c}\| = \alpha, \alpha \in \mathbb{R}$. These functions and their linear combinations are widely used for multidimensional interpolation and approximation of functions. This is due to their good approximating properties. Radial basis functions have great importance in the theory of RBF (radial basis function) of neural networks.

Consider the class of sums of radial basis functions

$$\mathcal{D} = \{r_1(\|\mathbf{x} - \mathbf{c}_1\| + r_2(\|\mathbf{x} - \mathbf{c}_2\| : \mathbf{x} \in Q, r_i \in C(\mathbb{R}), i = 1,2\},$$

where $Q \subset \mathbb{R}^d$ any compact set, $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed points, but functions r_i vary. The class $\mathcal{D}$ is the sum of algebras

$$A_1 = \{r_1(\|\mathbf{x} - \mathbf{c}_1\| : \mathbf{x} \in Q, r_1 \in C(\mathbb{R})\}$$

and

$$A_2 = \{r_2(\|\mathbf{x} - \mathbf{c}_2\| : \mathbf{x} \in Q, r_2 \in C(\mathbb{R})\}.$$

Consider the approximation of the function $f \in C(Q)$ by functions from D.

In this section, under certain restrictions (similar to the conditions of Theorem 9), the following formula is established for the approximation error:

$$E(f,\mathrm{D}) \overset{def}{=} \inf_{r\in \mathrm{D}} \| f - r \| = \sup_{p\subset Q} |G_p(f)|,$$

where the $\sup$ is taken over all closed paths with respect to centers $\mathbf{c}_1, \mathbf{c}_2$ of Q. Recall that path with respect to centers $\mathbf{c}_1$ and $\mathbf{c}_2$ defines as an ordered set of points $(\mathbf{p}_1, \mathbf{p}_2, ...) \subset Q$ with $p_i \neq p_{i+1}$, for which the equalities are satisfied $\|\mathbf{p}_1 - \mathbf{c}_1\| = \|\mathbf{p}_2 - \mathbf{c}_1\|, \|\mathbf{p}_2 - \mathbf{c}_2\| = \|\mathbf{p}_3 - \mathbf{c}_2\|, \|\mathbf{p}_3 - \mathbf{c}_1\| = \|\mathbf{p}_4 - \mathbf{c}_1\|, ...$ or $\|\mathbf{p}_1 - \mathbf{c}_2\| = \|\mathbf{p}_2 - \mathbf{c}_2\|, \|\mathbf{p}_2 - \mathbf{c}_1\| = \|\mathbf{p}_3 - \mathbf{c}_1\|, \|\mathbf{p}_3 - \mathbf{c}_2\| = \|\mathbf{p}_4 - \mathbf{c}_2\|, ...$. The path $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ with respect to centers $\mathbf{c}_1, \mathbf{c}_2$ is called closed, if the set $(\mathbf{p}_1, ..., \mathbf{p}_{2n}, \mathbf{p}_1)$ in this order is also a bolt with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$.

CONCLUSION

The following results were obtained in the dissertation work:

1) The necessary conditions are found for representing the space of continuous functions defined on a compact Hausdorff space as the sum of the subalgebras of this space. These conditions supplement the sufficient condition of Sproston and Straus for the possibility of such a representation.

2) It is proved that for the case of two algebras, the above necessary condition is also sufficient. The obtained result generalizes the classical Stone-Weierstrass theorem from the case of one algebra to the case of the sum of two algebras.

3) A necessary condition is found for the density of the sum of algebras in the space of continuous functions, which, under the corresponding restrictions, is also sufficient.

4) The convergence of the Diliberto-Straus algorithm to the approximation error in the problem of approximating a continuous function by the sum of two subalgebras of the space of continuous functions on a compact Hausdorff space is proved.

5) A necessary and sufficient condition is found for the element of the sum of algebras to be the best approximation (extreme element) for a given continuous function. The result obtained is an analogue of the Chebyshev theorem on characterizing a polynomial of best approximation.

6) The general results obtained are applied to the study of the approximating properties of the sum of ridge function algebras. A formula is established for calculating the error of approximation of a function of many variables by elements of the sum of two ridge algebras.

The results of dissertation are published in the following papers:

1. Asgarova, A. Kh. and Ismailov, V. E. A remark on the levelling algorithm for the approximation by sums of two compositions // - Baku: Caspian Journal of Applied Mathematics, Ecology and Economics, - 2016. 4 (1), - p. 30-37.
2. Asgarova, A. Kh. and Ismailov, V. E. Diliberto-Straus algorithm for the uniform approximation by a sum of two algebras // - India: Proceedings - Mathematical Sciences, Indian Academy of Sciences, - 2017. 127 (2), - p. 361-374.
3. Asgarova, A. Kh., Babayev, A. M-B., Maharov, I.K. On the error of approximation by ridge functions with two fixed directions // - Tbilisi: Tbilisi Mathematical Journal, - 2017. 10 (2), - p. 111-120.
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13. Asgarova, Aida Kh., Babayev, Arzu M-B., Maharov, Ibrahim K. // On the approximation by radial functions with fixed centers, International Conference “Modern Problems of Mathematics and Mechanics” devoted to the 60th anniversary of the Institute of Mathematics and Mechanics, - Baku: - 23 October - 25 October, - 2019, - p. 127-130.
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15. Askarova, A. Kh., Ismailov, V. E. A Chebyshev-Type Theorem Characterizing Best Approximation of a Continuous Function by Elements of the Sum of Two Algebras // - Russia: Математические Заметки, - 2021. 109 (1), - p. 19–26; Mathematical Notes, - 2021. 109 (1), - p. 15–20.

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Address: AZ 1141, Baku, B.Vahabzadeh, 9.

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