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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

SOLVABILITY OF SOMEBOUNDARY VALUE PROBLEMS IN A FINITE DOMAIN FOR OPERATOR COEFFICIENT THIRD ORDER EQUATIONS IN HILBERT SPACE

Speciality: 1202.01-Analysıs and functional analysis

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GENERAL CHARACTERISTICS OF THIS WORK

Rationale of the work and development degree.

The dissertation work was devoted to the solvability of boundary value problems on a finite segment for operator coeffcieint third order equations in Hilbert space and study of some spectral problems of operator pencils and homogeneous equation.

It is known that in some boundary value problems of mathematics, physics, partial differential equations and so on the study of various problems for operator-differential equations is one of the efficient methods. Note that starting with the works of E.Hille, R.Phillps, T.Kato and some other mathemticians the Cauchy problem was studied for first order equations. Then the Cauchy problem and boundary value problems were studied for second order and binomial equations. A part of the obtained results were reflected in the monographs of E.Hille, R.Phillipps, S.G.Krein, V.T.Gorbachuk, S.Y.Yakubov and Y.S.Yakubov and other authors.

Afterwards, the solvability of operator-differential eqautions was studied in infinite domains. As an example we can show the works of M.G.Gasymov, Y.A.Dubinsky, A.G.Kostyuchenko, A.A.Shkalikov, H.I.Aslanov, A.R.Aliyev, M.Bayramoglu and others.

Here we note the works of S.S.Mirzoyev, since he has found an efficients method for solving different boundary value problems. It should be noted that this method can not be applied to boundary value problems in a finite domain. Just for this reason, solvability conditions of boundary valu problems for only complete second order operator-differential equations were not studied by different methods. We can refer to these works the papers of S.S.Mirzoyev, G.A.Agayeva, M.Y.Salimov. For finding solvability conditions of complete equations it is necessary to estimate intermediate derivative operators. Unfortunately to day such exact estimations on a finite segment are not known. It should be noted that as we known a boundary value problem for third order binomial operator-differential equations on a finite segment was considered only in E.Obrechtin's work, but boundary value problems for complete operator-differential equaitons were not studied. In this work, the norms of intermediate derivative operators on a finite segment are estimated and solvability conditions of some boundary value problems for complete third order boundary value problems are found by their means.

One of the problems of thery of operator diferential equations is justification of solvability of the solution of a homogeneous equation by the Fourier method. This in its turn, leads to completeness of elementary solutions in the space of regular solutions. To obtain completeness of elementary solutions it is necessary to show some spectral properties of appropriate third order operator pencil, more exactly to show triple completeness of the system of eigen and associated vectors in the trace of regular solutions. Therefore, spectral theory of operator pencils is one of the rapidly developing fields of modern functional analysis.

Here we can indicate the works of M.V.Keldych, M.G.Gasymov, A.A.Shkalikov, G.V.Radzievsky, S.S.Mirzoyev and others.

Object and subject of research. Finding the conditions for solving three-order operator-differential equations on a finite section. Study of boundary value problems by Fredholm operator. Investigation of triple completeness of eigen and associated elements of operator groups according to operator-differential equations

The goal and objectives of the study.

1. Obtaining new theorems on solvability of boundary value problems on a finite segment for third order operator-differential equations.

2. Studying analytic properties of the resolvent of third order operator pencils and obtaining theorems on triple completeness of eigen and associated vectors in the trace spaces of regular solutions.

3. Obtaining new theorems on completeness of appropriate baundary value problem of the system of elementary solutions of a homogeneous equation in the space of regular solutions.

General technique of studies. In the work, the methods of theory of unbounded operators, semigroup theory of bounded operators, theory of complete functions in Hilbert space, theory of operator functions, theory of differential equations were used.

Main provisions of dissertation: The conditions of regular solvability of various boundary value problems for third order complete differential equations on a finite segment were found. These conditions were described by the equations of the coeffcieints of operator-differential coefficients.

The Fredholm property of the operators, obtained from perturbation of relatively complete continuous coeffcients differential equation of a complete operator differential equation on appropriate spaces was proved.

Triple completeness of eigen and associated elements of operator pencils corresponding to operator-differential equations was studied.

Completeness of the system of elementary solutions of homogeneous equation in the space of regular solutions was proved.

Scientific novelty.

1. New theorems on the regular and Fredholm solvability of some boundary value problems on a finite segment for third order complete operator differential equations were proved.

2. The norms of intermediate derivatives in Sobolev type space on a finite interval was estimated.

3. The relation between the solvability conditions of boundary value problems and the norms of intermediate derivative operators in a finite domain, was found.

4. New theorems on triple completeness of regular solutions of all eigen and associated vectors of third order operator pencils in the trace space were proved.

5. Solvability of boundary value problems for a homogeneous equation on a finite segment was studied and a theorem on completeness of a boundary value problem of the system of elementary solutions of a homogeneous equation in the space of regular solutions was proved.

Theoretical and practical value of the study. The works obtained in the dissertation work are of theoretical character, but the obtained results may be applied in some problems of partial differential equationsç of mathematical physics and mechanics.

Approbation and application. The main results of the dissertation were reported at the “Functional abalysis”, “Nonharmonic analysis” departments of IMM ANAS, at “Mathematical analysis”, “Differential and integral equations” chairs of Baku State University, at the International conference “Mathematical analysis, differential equations and their applications” (Baku 2015), International conference devoted to 80 years of Yahya Mammadov (Baku 2015), International conference “nonharmonic analysis and differential equations” (Baku 2016), International conference devoted to 80-th jubilee of acad. A.J.Hajiyev (Baku 2017).

Personal contribution of the author. All conclusions and results obtained belong to the author personally.

The name of the institution where the dissertation was completed. Institute of Mathematics and Mechanics of ANAS.

Publications. The results of the dissertation work were reflected in the applicant’s 7 scientific papers published in scientific editions recommended by the Higher Attestation Commission.Four of these papers are single-authered/Furthermore, the results obtained in the dissertation work have been reflected in the form of abstracts in 3 international level and 2 Republican level scientific conferences. One of them was published abroad.

Volume and structure of the dissertation. The dissertation work consists of 68 references. The total volume of the work consists of 241838 signs (title page -413 signs, contents – 1425 signs, introduction 34000 signs, chapter I – 124000 signs, chapter II – 82.000 signs).

DİSSERTATİON CONTENT

In the introduction of the dissertation work we justify its urgency, give works on the topic of the work, stated problems and their brief interpretation.

Assume that H is a separable Hilbert space, A is a self-adjoint positive operator. İt is known that the domain of definition $D(A^\gamma)$ of the operator A^γ $(\gamma \geq 0)$ is transformed to the Hilbert space with respect to the scalar product $(x,y)_\gamma = (A_x^\gamma, A_y^\gamma)$. $x,y \in D(A^\gamma)$. For $\gamma = 0$ we consider that $H_0 = H$.

Let $-\infty \leq a < b \leq +\infty$. By $L_2((a,b);H)$ we denote a Hilbert space of all vector-functions $f(t)$ almost everywhere defined on (a,b) with the values in H so that

$$\|f(t)\|_{L_2((a,b):H)}=\left(\int_a^b\|f(t)\|^2dt\right)^{1/2}<\infty .$$

Assume that $n=1,2,...$. Let us define the following Hilbert space

$$W_2^n((a,b);H)=\{u: A^n u\in L_2((a,b);H),\ u^{(n)}\in L_2((a,b);H)\}.$$

Here the norm is defined as

$$\|u\|_{W_2^n((a,b);H)}=\left(\|A^n u\|^2_{L_2((a,b);H)}+\|u^{(n)}\|^2_{L_2((a,b);H)}\right)^{1/2}$$

Here and in what follows, all derivatives are understood in the sense of a generalized derivative in abstract spaces

$a=-\infty,\ b=+\infty$, i.e. for $R=(-\infty,+\infty)$ as $L_2((-\infty,+\infty);H)=$

$=L_2(R;H)$ we denote, $W_2^n((-\infty,\infty);H)=W_2^n(R;H)$

For $n=3$ we define the following subspaces of $W_2^3((0,1);H)$

$W_2^3((0,1);H;\{0,1\},2)=\{u: u\in W_2^3((0,1);H),\ u(0)=0,\ u'(0)=0, u''(1)=0\}$,

$W_2^3((0,1);H;\{0\},1,2)=\{u: u\in W_2^3((0,1);H),\ u(0)=0, u'(1)=0, u''(1)=0\}$,

$\overset{0}{W}{}_2^3((0,1);H)=\{u: u\in W_2^3((0,1);H),\ u^{(k)}(0)=u^{(k)}(1)=0,\ k=\overline{0,2}\}$,

and for any real number α

$W_{2,\alpha}^3((0,1);H)=\{u: u\in W_2^3((0,1);H),\ u^{(k)}(0)=e^{i\alpha}u^{(k)}(1),\ k=\overline{0,2}\}$.

At first we study the problem

$$P(d/dt)u(t)=\frac{d^3u(t)}{dt^3}-A^3u(t)+\sum_{j=0}^{3}A_{3-j}u^{(j)}(t)=f(t),\ t\in(0,1), \quad (1)$$

$$u(0)=0,\ u'(0)=0,\ u''(1)=0 \quad (2)$$

Here the vector-functions $f(t), u(t)$ are almost everywhere defined on the interval $(0,1)$, their values are from H. The operator coefficients satisfy the following conditions.

1) A - is a self-adjoint positive definite operator;

2) The operator $B_j = A_j A^{-j}$ $(j=\overline{0,3})$ are bounded in H.

Definition 1. If for any $f(t) \in L_2((0,1);H)$ there exists such a vector-function $u(t) \in W_2^3((0,1);H)$ that almost everywhere satisfies equation (1) on (0,1), satisfies the boundary conditions (2) in the sense of convergence of

$$\lim_{t\to+0}\|u(t)\|_{\frac{5}{2}} = 0,\ \lim_{t\to+0}\|u'(t)\|_{\frac{3}{2}} = 0,\ \lim_{t\to+0}\|u''(1-t)\|_{\frac{1}{2}} = 0$$

and the estimation $\|u\|_{W_2^3((0,1);H)} \le const\|f\|_{L_2((0,1);H)}$ is valid, then problem (1),(2) is said to be regularly solvable.

At first we study regularity of the problem

$$P_0(d/dt)u(t) = \frac{d^3u(t)}{dt^3} - A^3u(t) = f(t),\ t \in (0,1), \tag{3}$$

$$u(0) = 0,\ u'(0) = 0,\ u''(1) = 0 \tag{4}$$

We prove the following theorem.

Theorem 1. Assume that condition 1) is satisfied. Then the operator P_0, where $P_0 u = u'''(t) - A^3u(t)$ isomorphically maps the space $W_2^3((0,1);H;(\{0,1\},2))$ into the space $L_2(0,1);H)$.

Hence we get that the boundary value problem (3),(4) is regularly solvable and in the space $W_2^3((0,1);H;(\{0,1\},2))$ the norm $\|P_0 u\|_{L_2((0,1);H)}$ is equivalent to the norm $\|u\|_{W_2^3((0,1);H)}$. Then according to the theorem on intermediate derivatives, the norms

$$N_k(\{0,1\},2) = \sup_{0\ne u\in W_2^3((0,1);H;(\{0,1\},2))} \left\|A^{3-k}u^{(k)}\right\|_{L_2((0,1);H)} \cdot \|P_0 u\|^{-1}_{L_2((0,1);H)},$$

$$k=\overline{0,3}$$

are finite. At first these norms are estimated and are related to the solvability of the problem (1),(2).

The following theorems are valid.

Theorem 2. The following estimations are valid for the norms $N_k(\{0,1\},2)$ ($k=\overline{0,3}$)

$$N_k(\{0,1\},2) \le c_k, \quad k=\overline{0,3},$$

so that $c_0 = c_3 = 1$, $c_1 = 2\cdot 3^{-1/2}$, $c_2 = 2^{1/2}\cdot 3^{-1/4}$.

Theorem 3. Assume that conditions 1), 2) are satisfied, and the equality

$$\gamma = \sum_{j=0}^{3} c_j \left\| B_{3-j} \right\| < 1, \ (B_j = A_j A^{-j}, \ j=\overline{0,3})$$

is satisfied. Here $c_0 = c_3 = 1$, $c_1 = 2\cdot 3^{-1/2}$, $c_2 = 2^{1/2}\cdot 3^{-1/4}$. Then problem (1), (2) is regularily solvable.

From this theorem we get the following result.

Result 1. Assume that all the conditions of theorem 3 are satisfied. Then problem

$$\frac{d^3u}{dt^3} + A^3u(t) + \sum_{j=0}^{3} A_{3-j}u^{(j)}(t) = f(t), \ t\in(0,1), \tag{5}$$

$$u''(0)=0, u(1)=0, \ u'(1)=0 \tag{6}$$

is regularly solvable.

Note that the definition of regular solvability of problem (5), (6) is similar to regular solvability of problem (1),(2).

We research Fredholm solvability of the problem

$$L(d/dt)u(t) = \frac{d^3u}{dt^3} - A^3u(t) + \sum_{j=0}^{2}(A_{3-j} + T_{3-j})u^{(j)}(t) = f(t), \ t\in(0,t), \tag{7}$$

$$u(0)=0, \ u'(0)=0, \ u''(1)=0 \tag{8}$$

In the space $W_2^3((0,1);H;\{0,1\},2)$ we define the operator $Lu = L(d/dt)u(t)$, $u \in W_2^3((0,1);H;\{0,1\},2)$ as L .

The following theorem is valid.

Theorem 4. Assume that A is a self-adjoint, positive-definite operator, has a completely continuous inverse A^{-1}. The operators $B_j = A_j A^{-j}$ $(j=\overline{1,3})$ are bounded in H, $T_j A^{-j}$ $(j=\overline{1,3})$ are completely continuous operators and the following inequality is valid:

$$q' = \sum_{j=0}^{2} c_j \left\| B_{3-j} \right\| < 1,$$

so, $c_0 = 1, c_1 = 2 \cdot 3^{-1/2}$, $c_2 = 2^{1/2} \cdot 3^{-1/4}$. Then the operator L is a Fredholm operator acting from the space $W_2^3((0,1);H;(\{0,1\},2))$ to the space $L_2((0,1);H)$.

Note that the analog of this theorem is valid for the problem (5),(6) as well.

Then we study regular solvability of the problem

$$P(d/dt)u(t) = \frac{d^3u(t)}{dt^3} + A^3u(t) + \sum_{j=0}^{3} A_{3-j}u^{(j)}(t) = f(t), \ t \in (0,1), \quad (9)$$

$$u(0) = 0, \ u'(1) = 0, \ u''(1) = 0 \quad (10).$$

Definition 2. If for any $f(t) \in L_2((0,1);H)$ there exists such a vector-function $u(t) \in W_2^3((0,1);H)$ satisfying equation (9) almost everywhere on (0,1) satisfies the boundary conditions in the sense of convergence (10) and the estimation

$$\lim_{t \to +0} \|u(t)\|_{\frac{5}{2}} = 0, \ \lim_{t \to +0} \|u'(1-t)\|_{\frac{3}{2}} = 0, \ \lim_{t \to +0} \|u''(1-t)\|_{\frac{1}{2}} = 0$$

$\| u \|_{W_2^3((0,1);H)} \le const \| f \|_{L_2((0,1);H)}$ is valid then problem (9),(10) is said to be regularly solvable

Here we introduce the space

$$W_2^3((0,1);H;\{0\},1,2) = \{u : u \in W_2^3((0,1);H), u(0) = 0, u'(1) = 0, u''(1) = 0\}$$

corresponding to the problem (9)-(10) and in this space we define the operator

$$P_0u = \frac{d^3u}{dt^3} + A^3u, \ u \in W_2^3((0,1);H;(\{0\},1,2))$$

and prove the following theorem.

Theorem 5. The operator P_0 isomorphically maps the space

$W_2^3((0,1);H;(\{0\},1,2))$ into the space $L_2((0,1);H)$.

Hence we get that the norms ,

$$N_k(\{0\},1,2)=\sup_{0\neq u\in W_2^3((0,1);H;(\{0\},1,2))}\left\|A^{3-k}u^{(k)}\right\|_{L_2((0,1);H)}\cdot\left\|P_0u\right\|_{L_2((0,1);H)}^{-1},$$

$(k=\overline{0,3})$ are finite.

The following theorem is proved.

Theorem 6. For the norms $N_k(\{0\},1,2)$ the followoing estimations are valid

$$N_k(\{0\},1,2)\leq\tilde{c}_k,\quad k=\overline{0,3},$$

so, $\tilde{c}_0=\tilde{c}_3=1,\ \tilde{c}_1=2^{2/3}\cdot3^{-1/2}$, $\tilde{c}_2=2\cdot3^{-1/2}$.

Using these estimations, we prove a theorem on regular solvability of the boundary value problem (9),(10).

Theorem 7. Assume that conditions 1), 2) are satisfied and the following inequality is valid

$$\gamma=\sum_{j=0}^{3}\tilde{c}_j\left\|B_{3-j}\right\|<1,$$

so that $\tilde{c}_0=\tilde{c}_3=1,\ \tilde{c}_1=2^{2/3}\cdot3^{-1/2}$, $\tilde{c}_2=2\cdot3^{-1/2}$. Then the problem (9),(10) is regularly solvable.

Hence we get the following result

Result 2. Assume that all the conditions of theorem 7 are satisfied

$$\frac{d^3u}{dt^3}-A^3u(t)+\sum_{j=0}^{3}A_{3-j}u^{(j)}(t)=f(t),\ t\in(0,1), \tag{11}$$

$$u'(0)=0,u''(0)=0,u(1)=0, \tag{12}$$

Then the problem $W_2^3((0,1);H;(\{0\},1,2))$ is regularly solvable The operator L is defined in the space as follows $Lu=L(d/dt)u(t)$, where,

$$L(d/dt)u(t)=\frac{d^3u}{dt^3}+A^3u+\sum_{j=0}^{2}(A_{3-j}+T_{3-j})u^{(j)}(t)\cdot$$

The following theorem is valid.

Theorem 8. Assume that conditions 1),2) are satisfied, A^{-1} is completely continuous, the operators $T_j A^{-1}$ $(j=\overline{1,3})$ are completely continuous in H and

$$\gamma' = \sum_{j=0}^{2} \tilde{c}_j \left\| B_{3-j} \right\| < 1,$$

such that ki, $\tilde{c}_0 = 1,\ \tilde{c}_1 = 2^{2/3} \cdot 3^{-1/2},\ \tilde{c}_2 = 2 \cdot 3^{-1/2}$. Then the operator L is a Fredholm operator acting from the space $W_2^3((0,1);H;\ (\{0\},1,2))$ to the space $L_2((0,1);H)$.

Then we study a periodic type problem. Let us consider such a problem

$$\frac{d^3u(t)}{dt^3} + A^3u(t) + \sum_{j=0}^{3} A_{3-j} u^{(j)}(t) = f(t),\ t \in (0,1), \tag{13}$$

$$u^{(k)}(0) = e^{i\alpha} u^{(k)}(1),\ k = \overline{0,2},\ \alpha \in R = (-\infty, \infty). \tag{14}$$

Note that for $\alpha = 2\pi k,\ k \in Z$ we get a periodic problem, for $\alpha = \pi(2k+1),\ k \in Z$ an antiperiodic problem.

Definition 3. If for any $f(t) \in L_2((0,1);H)$ there exists such a vector-function $u(t) \in W_2^3((0,1);H)$ satisfying equation (13) almost everywhere in (0,1) satisfies the boundary conditions (14) in the sense of convergence

$$\lim_{t \to +0} \left\| u^{(k)} - e^{i\alpha} u^{(k)}(1-t) \right\|_{3-k-\frac{1}{2}} = 0,\quad k = \overline{0,2}$$

and the estimation $\| u \|_{W_2^3((0,1);H)} \le const \| f \|_{L_2((0,1);H)}$ is valid, then the problem (13),(14) is said to be regularly solvable.

In the space

$$W_{2,\alpha}^3((0,1);H) = \{u : u \in W_2^3((0,1);H), u^{(k)}(0) = e^{i\alpha} u^{(k)}(1),\ k = \overline{0,2}\}$$

we consider the operator

$$P_0 u = \frac{d^3 u}{dt^3} + A^3 u .$$

At first we prove the following lemma.

Lemma 1. For any $u \in W_{2,\alpha}^3((0,1);H)$ the inequality

$$\|P_0 u\|_{L_2(0,1);H)} = \| u \|_{W_2^3((0,1);H)}$$

is valid.

Using this lemma, we prove the following theorem:

Theorem 9. The operator $P_0 : W_{2,\alpha}^2((0,1);H) \to L_2((0,1);H)$ is an isomorphism. Using this theorem, the norms of intermediate derivative operators

$$N_{k,\alpha} = \sup_{0 \neq u \in W_{2,\alpha}^3((0,1);H)} \left\| A^{3-k} u^{(k)} \right\|_{L_2((0,1);H)} \cdot \|P_0 u\|_{L_2((0,1);H)}^{-1},$$

$k = \overline{0,2}$ are estimated. The following theorem is valid.

Theorem 10. The norms $N_{k,\alpha}$ satisfy the following estimations:

$$N_{k,\alpha} \le p_k , \quad k = \overline{0,3} ,$$

so that, $p_0 = p_3 = 1,\ p_1 = p_2 = 2^{1/3} \cdot 3^{-1/2}$.

These estimations are related to the solvability conditions of the problem (13),(14).

Theorem 11. Assume that conditions, 1), 2) are satisfied and

$$\beta = \sum_{j=0}^{3} p_j \left\| B_{3-j} \right\| < 1 ,$$

so that, the numbers p_j $(j = \overline{0,3})$ are determined from theorem10. Then problem (13), (14) is said to be regularly solvable.

From this theorem we get that if the conditions of theorem 11 are satisfied, the periodic and antiperiodic problems are also regularly solvable. Then, it is shown that in the case of relative compact perturbation, the appropriate problem is of Fredholm type.

Chapter II of the dissertation work devoted to the study of resolvent properties of the operator pencil

$$P(\lambda) = \lambda^3 E - A^3 + \lambda^2 A_1 + \lambda A_2 + A_3 \qquad (15)$$

definition of the structure of spectrum, regular solvability of boundary value problems for a homogeneous equation, triple completeness of the system of eigen and associated vectors of the operator pencil (15), regular solutions in trace space and also to the proof of completeness of the elementary solutions of homogeneous equation

$$P(d/dt)u(t) = \frac{d^3u}{dt^3} - A^3u(t) + A_1\frac{d^2u}{dt^2} + A_2\frac{du}{dt} + A^3u = 0, \ t \in (0,1) \qquad (16)$$

in the space of regular solutions of some boundary value problems.

Let C be a complex plane. If for $\lambda \in C$ there exists such a $P^{-1}(\lambda)$ is bounded and determined on the whole of the space, then the point λ is said to be a regular point of $P(\lambda)$. Regular points are denoted as $\rho(P(\lambda))$. The set C $\rho(P(\lambda))$ is said to be a spectrum of $P(\lambda)$. So the function $P^{-1}(\lambda)$ is called a resolvents of $P(\lambda)$.

Definition 4. If $0 \neq x_{i,j,0} \in H_3$ satisfies the equation $P(\lambda_i)x_{i,j,0} = 0$, then the point λ_i is said to be an eigen point of $P(\lambda)$ while $x_{i,j,0}$ an eigen vector of $P(\lambda)$ corresponding to λ_i. If the system $\{x_{i,j,0},..., x_{i,j,m_{ij}}\}$

$$P(\lambda_i)x_{i,j,0} = 0, \qquad (x_{i,j,0} \neq 0),$$

$$P(\lambda_i)x_{i,j,1} + P'(\lambda_i)x_{i,j,0} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$P(\lambda_i)x_{i,j,m_{ij}} + P'(\lambda_i)x_{i,j,m_{ij}-1} + \frac{P''(\lambda_i)}{2!}x_{i,j,m_{ij}-2} + \frac{1}{3!}x_{i,j,m_{ij}-3} = 0$$

satisfies the equations then this system is said to the system of eigen and associated elements of $P(\lambda)$ corresponding to λ_i.

Definition 5. Assume that $\{x_{i,j,h}\}$, $j = \overline{1,q_i}$, $h = \overline{0,m_{i,j}}$ is the system of eigen and associated elements of $P(\lambda)$ corresponding to λ_i Then the vector functions

$$u_{i,j,h}(t) = e^{\lambda_i t}\left(\frac{t^h}{h!}x_{i,j,0} + \frac{t^{h-1}}{(h-1)!}x_{i,j,1} + \ldots + x_{i,j,h}\right), \; h = \overline{0, m_{ij}}$$

satisfy the equation $P(d/dt)u(t) = 0$ and are called elementary solutions of homogeneous equation (16).

At first we prove the discreteness of the spectrum of $P(\lambda)$ within certain conditions of the coefficients of $P(\lambda)$. More exactly, if

$1'$) A is a self-adjoint positive-definite operator and A^{-1} has a complete continuous inverse;

$2'$) The operators $B_j = A_j A^{-j}$ $(j = 1,2,3)$ are bounded in the space H;

$3'$) The operator $E + B_3$ has a bounded inverse in H and was defined on the whole of space.

Then the operator pencil $P(\lambda)$ has only a discrete spectrum, i.e. it consists of eigen numbers with finite iteration and their unique limit point is at infinity.

In the trace space $H_{5/2} \times H_{3/2} \times H_{1/2}$ we consider the system

$$K_0 = \{u_{i,j,h}(0), u'_{i,j,h}(0), u''_{i,j,h}(1)\}_{i=1}^{\infty}, \; j = \overline{1, q_i}, \; h = \overline{0, m_{ij}}$$

It is clear that the system K_0 is a system corresponding to the boundary value problem

$$P(d/dt)u = \frac{d^3u(t)}{dt^3} - A^3u(t) + \sum_{j=0}^{2} A_{3-j}\frac{d^j u^{(j)}}{dt^j} = 0, \; t \in (0,1), \quad (17)$$

$$u(0) = \varphi_0, \; u'(0) = \varphi_1, \; u''(1) = \varphi_2. \quad (18)$$

Definition 6. If the system K_0 is complete in the space $H_{5/2} \times H_{3/2} \times H_{1/2}$, we say that the system of eigen and associated vectors of the pencil $P(\lambda)$ is triple complete in the trace space of regular solutions.

At first we prove that subject to the conditions $1'$)-$3'$) a necessary and sufficient condition for the system of eigen and associated elements from a triple complete system in the trace space is obtaining from of the vector-function

$$R(\lambda)=(A^{5/2}P^{-1}(\overline{\lambda}))^{*}A^{5/2}\varphi+\lambda(A^{3/2}P^{-1}(\overline{\lambda}))^{*}A^{3/2}\psi+$$

$$+\lambda^{2}e^{\lambda}(A^{1/2}P^{-1}(\overline{\lambda}))A^{1/2}æ$$

on the plane C the factors the holomophy $\varphi\in H_{5/2},\psi\in H_{3/2},æ\in H_{1/2}$ $\varphi=\psi=æ=0$.

The following lemma is valid.

Lemma 2. Assume that conditions, $1'$) and $2'$) are fulfilled. If the inequality

$$2^{1/3}\cdot 3^{-1/2}\left(\|B_1\|+\|B_2\|\right)+\|B_3\|<1,$$

is satisfied, then there exists $P^{-1}(\lambda)$ in the imaginary axis and

$$|\lambda|^{3-\beta}\left\|A^{\beta}P^{-1}(\lambda)\right\|\le const,\ \ \beta\in[0,3]$$

Then we show that the clause of this lemma is valid also within very small sectors whose bisector is an imaginary axis.

Then we prove the following necessary lemma.

Lemma 3. Let conditions $0<\alpha<\pi/6$, 1), 2) be satisfied and

$$\sum_{j=0}^{2}d_{3,j}\left\|B_{3-j}\right\|<\sqrt{2}\sin\frac{3\alpha}{2},$$

so that $d_{3,3}=1,\ d_{3,1}=d_{3,2}=2^{1/3}\cdot 3^{-1/2}$. Then on the rays $\Gamma_{\alpha}=\{\lambda:\arg\lambda=\alpha\}$ then exists the resolvent $P^{-1}(\lambda)$ and on these rays the estimation

$$|\lambda|^{\beta}\left\|A^{3-\beta}P^{-1}(\lambda)\right\|\le const,\ \ \beta\in[0,3]$$

is valid.

Then we study regularity of problem (17),(18).

Definition 7. If for any $\varphi_0\in H_{5/2},\varphi_1\in H_{3/2},\varphi_2\in H_{1/2}$ the equation (16) has a regular solution and it satisfies the boundary conditions in the sense of convergence

$$\lim_{t\to+0}\left\|u(t)-\varphi_0\right\|_{5/2}=0,\ \lim_{t\to+0}\left\|u'(t)-\varphi_1\right\|_{3/2}=0,\ \lim_{t\to 1-0}\left\|u''(t)-\varphi_2\right\|_{1/2}=0$$

and the estimation

$$\| u \|_{W_2^3((0,1);H)} \le const\left(\left\|\varphi_0\right\|_{5/2} + \left\|\varphi_1\right\|_{3/2} + \left\|\varphi_2\right\|_{1/2} \right),$$

is valid, then problem the problem (17),(18) is called regular solvable.

The following theorem is valid.

Theorem 12. Assume that all the conditions of theorem 3 are satisfied. Then the problem (17),(18) is regularly solvable.

Similar theorems are valid for other boundary value problems as well.

In this chapter, we give the notion of inner compactness of the space of regular solutions of the equation $P(d/dt)u(t)=0$.

Definition 8. Assume that $L(p)=KerP(d/dt)=\{u: u\in W_2^3((0,1);H),\quad P(d/dt)u(t)=0\}$. $0\le a<a'<b'<b\le 1$ and $M>0$ are arbitrary numbers. İf the set

$$L_M=\{u: u\in L(P),\ \|u\|_{W_2^2((a,b);H)}\le M\}$$

is compact with respect to the norm $\|u\|_{W_2^2((a_1,b_1);H)}$ then we say that the space of regular solutions of the homogeneous equation is inner compact.

We prove the following theorem

Theorem 13. Assume that conditions 1),2) are satisfied, A^{-1} is completely continuous. If the inequality

$$\sum_{j=0}^{2} d_{3,j}\left\|B_{3-j}\right\|<1$$

is satisfied, then the space of regular solutions of the homogeneous equation is inner compact . Here $d_{3,0}=1,\ d_{3,1}=d_{3,2}=2^{1/3}\cdot 3^{-1/2}$.

In hits chapter, we prove the main theorem on triple completeness of eigen and associated elements of the operator pencil $P(\lambda)$ in the trace space.

Theorem 14. Assume that A is a self-adjoint, positive-definite operator, $A^{-1} \in \sigma_\rho$ $(0 < \rho < \infty)$, $B_j = A_j A^{-j}$ $(j = \overline{1,3})$ are bounded in H and

$$q = \sum_{j=0}^{2} c_j \left\| B_{3-j} \right\| < \begin{cases} 1, & 0 < \rho \le 3 \\ \sqrt{2} \sin \dfrac{3\pi}{4\rho}, & 3 \le \rho < \infty, \end{cases}$$

so that $c_0 = 1,\ c_1 = 2 \cdot 3^{-1/2}, c_2 = 2^{-1/2} \cdot 3^{-1/4}$. Then the system of eigen and associated vectors of the operator pencil $P(\lambda)$ is triple complete in the trace space of regular solutions.

Using the theorem on regular solvability of boundary value problem (17),(18) and theorem 14 we get the following theorem on the completeness of the system of elementary solutions of the homogeneous equation.

Theorem 15. Assume that all the conditions of theorem 14 are satisfied. Then the system of elementary solutions of the equation $P(d/dt)u(t) = 0$ is complete in the space of regular solutions.

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CONCLUSION

The dissertation is devoted to the solution of some boundary value problems in the finite domain for equations with three-order operator coefficients in Hilbert space.

The following main results were obtained in the dissertation:

1. New theorems on the regular and fredholm solution of some boundary value problems in the finite domain for third-order complete operator differential equations.
2. In the finite interval, the norms of intermediate derivatives in Sobolev-type spaces were evaluated.
3. A connection has been found between the conditions for resolving boundary issues and the norms of intermediate derivative operators in a finite domain.
4. New theorems on the triple completeness of all eigenvalues and conjugation vectors of three-order operator sets in the trace space of regular solutions.
5. The solution of boundary value problems in a finite section for a homogeneous equation is studied, and the theorem on the completeness of the system of elementary solutions of a homogeneous equation in the space of regular solutions of the boundary value problem is proved.

The main results of the dissertation were published in the following works.

1. Мирзоев, С.С., Гейдарова, С.Б. О разрешимости одной краевой задачи для операторно-дифференциальных уравнений третьего порядка на конечном отрезке // Proceedings of IAM, -2015. v. 4, № 1, -с. 26-39.
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