REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

THE SPECTRAL PROPERTİES OF A DİSCONTİNUOUS DİFFERENTİAL OPERATOR WİTH A SPECTRAL PARAMETER İN BOUNDARY CONDİTİON

Speciality: 1202.01- Analysis and functional analysis

Field of science: Mathematics

Applicant: **Gular Vidadi kizi Maharramova**

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

Baku – 2021

The work was performed at the department of “Non-harmonical analysis” Institute of Mathematics and Mechanics Azerbaijan National Academy of Science

Scientific supervisor: dr. math. sci., assoc. prof

Telman Benser oglu Gasymov

Official opponents: dr. of ph. and math. sci., prof.

Hamidulla Israfil oglu Aslanov

dr. of ph. and math. sci., prof.

Ibrahim Mayil oglu Nabiyev

ph.d. in math.

Ali Abbas oglu Huseynli

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan

Chairman of the Dissertation council: Corr. Member of ANAS,

dr. of ph. and math. sci., prof.

__________ **Misir Jumail oglu Mardanov**

Scientific secretary of the Dissertation council:

cand. of ph. and math. sci.

_________ **Abdurrahim Farman oglu Guliyev**

Chairman of the scientific seminar: Corr. member of ANAS,

dr. of ph. and math. sci., prof.

____________ **Bilal Telman oğlu Bilalov**

GENERAL CHARACTERISTICS OF WORK

Relevance and degree of development of the topic. Many problems in modern mathematics, mechanics and physics require new approaches to solving partial derivative differential operators that describe certain physical processes. In most cases the method of separation of variables is used. The solution of special derivative differential equations by the Fourier method yields to the study of the spectral properties of the corresponding differential operators. This problem includes the completeness, minimality and basis properties of the system of eigenfunctions and associated functions of an ordinary differential operators in different functional spaces.

The spectral properties of problems with a spectral parameter in boundary condition have been studied in the works of Rasulov M.L., Walter J., Fulton C.T., Schneider A., Russakovsky E.M., Hinton D.B., Dijksma A., Binding P.A., Browne P.J., Watson B.A. In the paper of Shkalikov A.A. was constructed the general theory for an ordinary differential operator when the boundary condition and the coefficients of equation polinomially depends on spectral parameter. In the paper Shkalikov A.A.[1] was contructed the general theory for an ordinary differential operator when the boundary condition and the coefficients of equation polinomially depends on spectral parameter. In the paper of Moiseyev Y.I and Kapustin N.Y.[2] was shown for the Shturm-Liouville problem with a spectral parameter in boundary condition that, the system of eigenfunctions of problem forms a Riesz basis for space L_2 if we eliminate any eigenfunction. Different generalizations of this results were obtained in the works of Moiseyev Y.I., Kapustin N.Yu., Karimov N.B., Mirzoyev V.S.,

---

[1] Shkalikov *A.A.,* Boundary value problems for ordinary differential equations with the parameter in the boundary condition // -Moscow: Proceedings of I.G.Petrovskiy seminars, - 1983, т. 9, -p. 190-229.

[2] Kapustin N.Yu., Moiseev E.I. On spectral problems with spectral parameters in the boundary condition //- Moscow: Differential equations, -1997. т.33, № 1,- p. 115-119.

Aliyev Z.S., Poladov R.G., Marchenkov D.B.In more general terms the abstract analogues of this problem was considered in the work of Bilalov B.T. and Muradov T.R..[3] Before this work Gohberg I.S., Markus A.S., Shkalikov A.A. studied defect basis. In the work of Gasymov T.B.[4] this problems were studied in detail in the abstract and contructive necessary and sufficient conditions were found. In the paper of Gasymov T.B and Mammadova Sh.J.[5], Gasymov T.B. and Huseynli A.A.[6] the spectral problem which has a discontinuity point in the middle and with spectral parameter in boundary condition have been studied by resolvent method. In the papers of Bilalov B.T. and Gasymov T.B.[7] introduced a new method for studying the basis properties of discontinuous differential operators. From the recent works Shkalikov A.A.'s[8] work emphasizes the importance of an abstract approach and explains its wide applications. Related to the basicity of eigebnfuncions of discontinuous differential operators in Lebesgue spaces we can note the works of Gurbanov V.M and his students, where are studied the basicity generalized eigenfunctions and associated functions in the

---

[3] Bilalov B.T., Muradov T.R. Defective bases of Banach spaces // Proc. IMM NAS Azerb., 2005, v.22, pp. 23-26.

[4] Gasymov T.B. On necessary and sufficient conditions of basicity of some defective systems in Banach spaces // Trans. NAS Azerb., ser. phys.-tech. math. sci., math.mech., 2006, v.26, №1, p.65-70.

[5] Gasymov T.B., Mammadova Sh.J. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary condition // Trans. Of NAS of Azerb. 2006,vol. XXVI, №4, p. 103-116.

[6] Gasymov T.B., Huseynli A.A. The basis properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Proc. of IMM of NAS of Azerb. vol. XXXV(XLIII), 2011, pp. 21-32.

[7] Bilalov B.T., Gasymov T.B. On bases for direct decomposition. Doklady Mathematics. 93(2) (2016).pp 183-185.

[8] Shkalikov A.A., On basis properties of root functions of differential operators with spectral parameter in the boundary condition //- Moscow: Differential equations,-2019. т.55, №5,- p.647-659.

sense of V.A.Ilin. One or another spectral properties of discontinous differential operators were studied in the works of Lomov I.S, Gomilko A.M. and Pivovarchik V.N., Shahriari M., Shahriari M., Jodayree, Akbarfam A. and Teschli G.

In the dissertation work is given a general approach for the studying of discontinuous differential operators with a spectral parameter in boundary condition, abstract theorems for the investigating basicity properties of the system of eigenfunctions and associated functions of discontinuous differential operators and is given its application to the second order discontinuous differential operators are proved. So that, in the work was studied the basicity of the systems of eigenfunctions and associated functions of the second order discontinuous differential operator with a spectral parameter in boundary condition in Morrey and Lebesgue type spaces with the help of abstract theorems. Considered spectral problems are regular for Birkhof's classification, but they are not strongly regular. The study of basis properties such spectral problems with resolvent method is accompanied by certain difficulties. Such types spectral problems was studied in the works of Mammadov Kh.R. and Karimov N.B., Dernek N. and Veliev O.A., Djakov P. and Mityagin, B. Therefore, we think that the theme of the dissertation is actual and evokes scientific interest.

Object and subject of research. The direct sum of Banach spaces, discontinuous differential operators.

The goal and objectives of the study. The main goal of this work is the studying the basis properties of eigenfunctions and associated functions of a discontinuous second order differential operator with a spectral parameter in boundary condition in $L_p \oplus C$, L_p Lebesgue spaces and $M^{p,\alpha} \oplus C$, $M^{p,\alpha}$ Morrey type spaces.

General technique of studies. In this work are applied the methods of the theory of real and complex variable functions, the methods of the differential operators, the methods of functional analysis, also the methods of the theory of linear operators in Hilbert and Banach spaces, the methods of the theory of bases and frames, the methods of the theory of approximation.

Main provisions of dissertation.

The first chapter devoted the basis properties of systems in the direct sum of Banach spaces. In this chapter is given an abstract approach, is studied the direct decomposition of Banach spaces, a new method is proposed for studying the basicity of the whole space from the basicity of its subspaces.

In second chapter is considered the spectral problem with a spectral parameter in boundary condition. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying Fourier method. The basicity of eigenfunctions of spectral problem in spaces $L_p \oplus C$ and L_p is studied in this chapter.

The third chapter devoted the studying of basicity of eigenfunctions of spectral problems in Morrey type spaces.

Scientific novelty. In the work is obtained the following scientific novelties:

-based on the corresponding systems of subspaces in the direct sum of Banach spaces the theorems about the completeness, minimality, basicity, also *p*-basicity of systems were proved;

-the asymptotics of the eigenvalues and eigenfunctions of the discontinuous second order differential operator with spectral parameter in boundary condition was found;

-the resolvent of the linearizing operator of considering spectral problem was constructed, in $L_p \oplus C, L_p, 1 < p < +\infty$ Lebesgue spaces, theorems about the completeness and minimality of the system of eigenfunctions and associated functions of linearizing operator were proved;

-theorems about the basicity of eigenfunctions and associated functions of discontinuous second order differential operator in $L_p \oplus C$ and $L_p, 1 < p < +\infty$, Lebesgue spaces, and theorems about the Riesz basicity in $L_2 \oplus C$ və L_2 spaces were proved;

- theorems about the basicity of eigenfunctions and associated functions of discontinuous second order differential operator in $M^{p,\alpha} \oplus C$ və $M^{p,\alpha}, 1 < p < +\infty, 0 < \alpha \leq 1$ Morrey type spaces were proved.

Theoretical and practical value of the study. The results of the dissertation are theoretical. They can be used in the spectral theory of differential operators, in the solution of various problems of mathematical physics and mechanics, including the theory oscillations, hydromechanics, to substantiate by the Fourier method of the solution of the problems of plasticity theory leading to the studying of non-selfadjoint diffrential operators. The results can also be used in theory of approximation theory.

Approbation and application. The main results of the dissertation were presented and discussed at the following seminars, republican and international scientific seminars: At the seminars of "Non-harmonical analysis" department of IMM of ANAS (the head of the department corr. Member of ANAS, prof. B.T.Bilalov), at the seminars of "The theory of functions and functional analysis" chair of BSU (the head of the department prof. A.Ahmadov) as well as at the international scientific conference " On Actual Problems of Mathematics and Informatics", dedicated to the 90-th anniversary of Haydar Aliyev (Baku, 2013), at the republic scientific conference "Problems of application of mathematics " (Baku, 2013), at the republican scientific conference "Functional analysis and its applications" dedicated to the 100-th anniversary of prof. A.Habibzadeh (Baku, 2016), at the international scientific seminar organized by the Institute of Mathematics and Mechanics of ANAS " Non-harmonical analysis and differential operators" (Baku, 2016), at the international scientific conference "Differential equations and mixed problems" (Sterlitamak, 2018), at the international scientific conference "The spectral theory and mixed problems" (Ufa, 2018), at the international scientific conference dedicated to the 90-th anniversary of V.A.Ilin, (Russia, 2018), at the republican scientific conference "Mathematics, Mechanics and their applications“ dedicated to the 97-th anniversary of Haydar Aliyev (Baku, 2020).

Personal contribution of the author. All the results obtained in the work are the personal contribution of the author.

Publications of the author. The main results of the work- 8 of them were published in the recommended journals of the EAC under the President of the Republic of Azerbaijan, 2 of them without co-author, including 4 in periodicals published in the international summarization and indexing systems. Published on the results of national and international scientific events are 10 (3 of them were published abroad).

The name of the institution where the dissertation was completed. The work was performed at the Non-harmonical analysis department of Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan.

Volume and structure of the dissertation (in signs, indicating the volume of each structural unit separately). The dissertation consists of introduction -41135 characters (title page and content -2998 characters), 3 chapters- (I chapter-39574 character, II chapter-118807 character, III chapter-33425 character) and the list of used literature consists of 94 items. The total volume of work is 146 pages.

THE CONTENT OF THE DISSERTATION

Dissertation consists of introduction, three chapters, conclusions and the list of used literature.

The first chapter is devoted to the basis properties of systems in the direct sum of Banach spaces. In this chapter an abstract approach is given, the direct decomposition of Banach space is studied, a new method is proposed for studying the basis of the whole space from the basis of its subspaces. This approach has wide applications in the spectral theory of discontinuous differential operators. İn this chapter the completeness, minimality and basis property of the system $\hat{u}_{in} = \left(a_{i1}^{(n)} u_{1n}, \ldots, a_{im}^{(n)} u_{mn}\right)$, $i = \overline{1,m}, n \in N$ is investigated, and obtained some results during the direct decomposition $X = X_1 \oplus \ldots \oplus X$.

In section 1.1 the necessary information is given, the main consepts and facts of basis theory which used in the dissertation is shown.

Statement 1. *Let the system $\{x_n\}_{n\in N}$ forms a basis with parantheses for a space X and $\{n_k\}_{k\in N}$ is a sequence of indexes. If the system $\{x_n\}_{n\in N}$ is a uniform minimal system and if the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded, then this system forms a usual basis for X .*

Definition 1. *The basis which equivalent to the orthonormal basis in H Hilbert space is called Riesz basis.*

Definition 2. *Let the system $\{x_n\}_{n\in N}$ is minimal in X Banach space and the system $\{x_n^*\}_{n\in N}$ is its biorthogonal system. If there is a constant M , which for an arbitrary $x \in X$ the following inequality*

$$\left(\sum_{n=1}^{\infty} \left| \left\langle x, x_n^* \right\rangle \right|^p \right)^{\frac{1}{p}} \le M \|x\|$$

holds, then the system $\{x_n\}_{n\in N}$ is called a p-Bessel system . If the basis is a p-Bessel system, then it called a p-basis.

Definition 3. *The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ of Banach space is called a p-close,if*

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^p < \infty .$$

In special case, if p=2 they called a square close.

Statement 2. *Let the system $\{u_n\}_{n\in N}$ forms a Riesz basis with parentheses for the Hilbert space H , and the system $\{\vartheta_n\}_{n\in N}$ is biorthogonal system. If the condition $\sup_n \{\|u_n\|; \|\vartheta_n\|\} < +\infty$ holds and* *the sequence* $\{n_{k+1}-n_k\}_{k\in N}$ *is bounded, then the system* $\{u_n\}_{n\in N}$ *forms a usual Riesz basis for the space* H. *Where the system* $\{n_k\}_{k\in N}$ *is the system of indexes in the definition of the basis with parantheses.*

In the section 1.2 the completeness and minimality of the systems is investigated for the direct sum of Banach spaces and some theorems are proved.

Let $X = X_1 \oplus \ldots \oplus X_m$ holds, where $X_i, i = 1,2,\ldots,m$ -is an arbitrary Banach space and for each space $X_i, i = 1,2,\ldots,m$ is given an arbitrary system $\{u_{in}\}_{n\in N}$. Let consider the following system in the space X .

$$\hat{u}_{in} = \left(a_{i1}^{(n)}u_{1n},\ldots,a_{im}^{(n)}u_{mn}\right), i = \overline{1,m}, n \in N \qquad (1)$$

where $a_{ij}^{(n)}$ -are arbitrary numbers. Let $A_n = \left(a_{ij}^{(n)}\right)_{i,j=\overline{1,m}}$; $\Delta_n = \det A_n$. The following theorems are true:

Theorem 1. *Let the system* $\{u_{in}\}_{n\in N}$ *be a complete (minimal) system in* $X_i, i = \overline{1,m}$. *If* $\Delta_n \neq 0, \forall n \in N$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is also complete (minimal) in* X .

Theorem 2. *Let the system* $\{u_{in}\}_{n\in N}$ *be a minimal system in* $X_i, i = \overline{1,m}$. *If* $\exists n_0 \in N, \Delta_{n_0} = 0$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not minimal in* X .

Theorem 3. *Let the system* $\{u_{in}\}_{n\in N}$ *be complete and minimal system in* X_i, *for each* $i \in \overline{1,m}$. *If* $\exists n_0 \in N, \Delta_{n_0} = 0$, *then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *is not complete and is not minimal in* X .

In section 1.3 was studied the basicity, *p*-basicity of systems in the direct sum of Banach spaces, including Riesz basis in the direct sum of Hilbet spaces.

Theorem 4. *Let the system* $\{u_{in}\}_{n\in N}$ *forms a basis for* X_i *for each* $i \in \overline{1,m}$. *If all* $\Delta_n = \det\left(a_{ij}^{(n)}\right) \neq 0, n \in N$, *then the system*

$\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *forms a basis with parentheses in* $X = X_1 \oplus \ldots \oplus X_m$. *If, in addition the conditions*

$$\sup_n \left\{\|A_n\|, \left\|A_n^{-1}\right\|\right\} < +\infty, \ \sup_n \left\{\|u_{in}\| : \|\vartheta_{in}\|\right\} < +\infty, \ i = \overline{1,m}, \qquad (2)$$

also hold, then the system $\{\hat{u}_{in}\}_{i=\overline{1,m};n\in N}$ *forms a usual basis in* X .

The analogue of this theorem in the case of Hilbert space for Riesz basis will be as following theorem.

Theorem 5. *Let* X_i *be Hilbert space and the system* $\{u_{in}\}_{n\in N}$ *forms a Riesz basis for each* X_i, $i = \overline{1,m}$ *space. Then under condition* $\Delta_n \neq 0, n \in N$, *the system* $\{\hat{u}_{in}\}_{i=\overline{1,m}:n\in N}$ *forms a Riesz basis with parentheses for* $X = X_1 \oplus \ldots \oplus X_m$. *If in addition, the conditions (2) hold, then the system* $\{\hat{u}_{in}\}_{i=\overline{1,m}:n\in N}$ *forms a usual Riesz basis for* X .

Let $J = \{n_1,\ldots,n_m\}$ be some set of m natural numbers. In section 1.3 was proved anologues of some theorems above in the case of *p*-basis. Let X be a Banach space and the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ is some system in X .

Let $a_{ik}^{(n)}, i,k = \overline{1,m}, n \in N$ be some complex numbers. Let $A_n = \left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}}$ and $\Delta_n = \det A_n$, $n \in N$. Consider the following system in X

$$\hat{u}_{kn} = \sum_{i=1}^{m} a_{ik}^{(n)} u_{in}, \quad k = \overline{1,m}; n \in N$$

Theorem 6. *Let the system* $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ *forms a p-basis in* X *and the system* $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N} \subset X^*$ *is its biorthogonal system. If the condition* $\Delta_n \neq 0, \forall n \in N$ *holds, then the system* $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ *forms a p-basis with parentheses in* X . *If the system* $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ *is p-basis and in addition the conditions (2) hold, then the system* $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ *also forms p-basis for* X .

Let X be a Banach space, $X_1 = X \oplus C^m$ and the system $\{\hat{u}_n\}_{n\in N} \subset X_1$ is some minimal system, $\{\hat{\vartheta}_n\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ is its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1},...,\alpha_{nm}); \quad \hat{\vartheta}_n = (\vartheta_n; \beta_{n1},...,\beta_{nm})$$

Let

$$\delta = \det \left\| \beta_{n_i j} \right\|_{i,j=\overline{1,m}} \tag{3}$$

Theorem 7. *Let the system* $\{\hat{u}_n\}_{n\in N}$ *forms a p-basis for* X_1*. If the determinant* $\delta \neq 0$ *which determined by formula (3), then the system* $\{u_n\}_{n\in N_J}$ *,* $N_J = N \setminus J$ *forms a p-basis for* X *. In this case the biorthogonal system of the system* $\{u_n\}_{n\in N_J}$ *determined by the following formula:*

$$\vartheta_n^* = \frac{1}{\delta} \begin{vmatrix} \vartheta_n & \vartheta_{n1} & \cdots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \cdots & \beta_{n_m 1} \\ \cdots & \cdots & \cdots\cdot & \cdots \\ \beta_{nm} & \beta_{n_1 m} & \cdots & \beta_{n_m m} \end{vmatrix}$$

In the second chapter we consider the following spectral problem:

$$l(y) = -y'' + q(x)y = \lambda y, \quad x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right) \tag{4}$$

$$\left. \begin{aligned} & y(0) = y(1) = 0, \\ & y(\tfrac{1}{3} - 0) = y(\tfrac{1}{3} + 0), \\ & y'(\tfrac{1}{3} - 0) - y'(\tfrac{1}{3} + 0) = \lambda m y(\tfrac{1}{3}) \end{aligned} \right\} \tag{5}$$

where λ is a spectral parameter, *m* is a non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method.

In section 2.1 was constructed the linearizing operator and Green's function for this spectral problem. The Green's function is determined as the satisfying (5) boundary conditions and the kernel of the integral expressions of solution of the following non-homogeneus equation

$$-y''(x)+q(x)y-\lambda y(x)=f(x).$$

Denote by $W_p^k\left(0,\frac{1}{3}\right)\oplus W_p^k\left(\frac{1}{3},1\right)$ the space of functions whose restrictions to intervals $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},1\right]$ belong to Sobolev spaces $W_p^k\left(0,\frac{1}{3}\right)$ və $W_p^k\left(\frac{1}{3},1\right)$ respectively. Let us define the operator L in the space $L_p(0,1)\oplus C$ in the following way:

$$D(L)=\left\{\hat{u}\in L_p(0,1)\oplus C:\hat{u}=\left(u(x);\ mu\left(\frac{1}{3}\right)\right),\ u\in W_p^2\left(0,\frac{1}{3}\right)\oplus\right.$$
$$\left.\oplus W_p^2\left(\frac{1}{3},1\right), u(0)=u(1)=0, u\left(\frac{1}{3}-0\right)=u\left(\frac{1}{3}+0\right)\right\} \tag{6}$$

and for $\hat{u}\in D(L)$

$$L\hat{u}=\left(l(u);\ u'\left(\frac{1}{3}-0\right)-u'\left(\frac{1}{3}+0\right)\right),\quad \hat{u}\in D(L) \tag{7}$$

Teorem 8. *The operator L defined by expressions (6)-(7) is a densely defined closed operator with completely continuous resolvent. The eigenvalues of the operator L and the problem (4)-(5) coincide. If $u(x)$ is the eigenfunction (associated function) of problem (4)-(5), then $\hat{u}=\left(u(x);\ mu\left(\frac{1}{3}\right)\right)$ is the eigenvector (associated vector) of the operator L.*

Let $\hat{f} = (f, \beta) \in L_p(0,1) \oplus C$ and consider the following equation:

$$L\hat{u} = \lambda\hat{u} + \hat{f} \tag{8}$$

Denote by $\Delta(\lambda)$ the characteristic determinant of the spectral problem (4)-(5). In this section we obtained an integral expression for the solution of equation (8), which allows to determine the structure of Green's function and resolvent operator. Here also determined the maximal and minimal operators which evoke by differential expression and at the result proved the following theorem about the solvoability of the equation (8).

Theorem 9. *There is a unique solution of the equation (8) for* $\forall\hat{f} \in L_p(0,1) \oplus C$ *in each value of parameter* λ *satisfying the condition* $\Delta(\lambda) \neq 0$ $\forall\hat{f} \in L_p(0,1) \oplus C$ $\hat{u} = \left(u, mu\left(\frac{1}{3}\right)\right)$. *The eigenfunctions of the operator* L *may be zeros only* $\Delta(\lambda)$ *and that is why it is no more counting. The limit point of eigenvalues only can be in infinity. The root vectors of the operator* L *operatorunun root vectors are minimal in* $L_p(0,1) \oplus C$ *space.*

In the section 2.2 the spectral problem (4)-(5) is studied in special case, we mean $q(x) \equiv 0$.

$$y'' + \lambda y = 0, \quad x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \tag{9}$$

$$\left.\begin{aligned} &y(0) = y(1) = 0, \\ &y\left(\frac{1}{3} - 0\right) = y\left(\frac{1}{3} + 0\right), \\ &y'\left(\frac{1}{3} - 0\right) - y'\left(\frac{1}{3} + 0\right) = \lambda m y\left(\frac{1}{3}\right), \end{aligned}\right\} \tag{10}$$

where λ is a spectral parameter, m is a non-zero complex number.

Theorem 10. *The spectral problem (9), (10) has two series of eigenvalues:*

$$\lambda_{1,n} = (\rho_{1,n})^2,\ n = 1,2,\ldots,\ v\text{ə}\ \lambda_{2,n} = (\rho_{2,n})^2,\ n = 0,1,2,\ldots$$

where

$$\left.\begin{aligned} \rho_{1,n} &= 3\pi n, \\ \rho_{2,n} &= \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O(\frac{1}{n^2}). \end{aligned}\right\}$$

The corresponding eigenfunctions $u_n(x),\ n = 0,1,2,\ldots,$ *are given by the following expressions*

$$\begin{cases} u_{2n-1}(x) = \sin 3\pi m x, \qquad x \in [0,1], \quad n = 1,2,\ldots \\ u_{2n}(x) = \begin{cases} \{\sin \rho_{2,n}(x - \frac{1}{3}) + \sin \rho_{2,n}(x + \frac{1}{3}), & x \in \left[0, \frac{1}{3}\right], \\ \sin \rho_{2,n}(1 - x), & x \in \left[\frac{1}{3}, 1\right], n = 0,1,2,\ldots \end{cases} \end{cases}$$

The Green's function of problem (9),(10) is defined as a kernel of integral representation for solution of the corresponding non-homogeneus problem

$$y''(x) + \rho^2 y(x) = f(x) \tag{11}$$

satisfying boundary condition (10).

Theorem 11. *The system of eigenvectors of the operator* L *(the linearizing operator of the spectral problem (9)-(10)) is complete in* $L_p(0,1) \oplus C, \quad 1 < p < \infty$.

Corollary 1. *The system of eigenvectors of the operator* L $\{\hat{u}_n\}_{n \in N_0}$ *is complete and minimal in* $L_p(0,1) \oplus C, 1 < p < \infty$.

Theorem 12. *If* $n_0 \in N_0$ *an even number and corresponding eigenvalue is simple, then the system* $\{u_n(x)\}_{n \in N_0 \setminus \{n_0\}}$ *is complete and minimal in* $L_p(0,1), 1 < p < \infty$. *If* n_0 *is an odd number, then the* *system* $\{u_n(x)\}_{n\in N_0\setminus\{n_0\}}$ *is not complete and is not minimal in this space.*

Theorem 13. *The eigenvectors of operator* L $\{\hat{u}_n\}_{n\in N_0}$ *forms basis in* $L_p(0,1)\oplus C$ *when* $1<p<\infty$, *and forms Riesz basis when* $p=2$.

Theorem 14. *If from the system of eigen and associated functions of problem (9)-(10)* $\{y_0\}\cup\{y_{i,n}\}_{i=1,2;\,n\in N}^{\infty}$ *we eliminate any function* $y_{2,n_0}(x)$, *corresponding to a simple eigenvalue, then the obtaining system foms a basis for* $L_p(0,1), 1<p<\infty$, *and forms a Riesz basis for* $L_2(0,1)$. *If we eliminate any function* $y_{1,n_0}(x)$ *from this system, then the obtaining system does not form a basis in* $L_p(0,1)$, *moreover, in this case the obtained system is not complete and is not minimal in this space.*

In the section 2.3 the basicity of eigenfunctions was studied in the case when potential was introduced. For expressing the results let us introduce the following functions:

$$q_1(x)=\frac{1}{2}\int_0^x q(t)dt\,,\ q_2(x)=\frac{1}{2}\int_x^1 q(t)dt\,.$$

For the eigenvalues and eigenfunctions of problem (4)-(5) proved the following theorems.

Theorem 15. *The spectrum of problem (4)-(5) consists of three sequences* $\lambda_{i,n}=\rho_{i,n}^2, i=1,2,3;\ n=1,2,...,$ *of asymptotically simple eigenvalues:*

$$\begin{cases} \rho_{1,n} = 3\pi n + O\left(\dfrac{1}{n^2}\right), \\ \rho_{2,n} = 3\pi n + \dfrac{\alpha_1}{n} + O\left(\dfrac{1}{n^2}\right), \\ \rho_{3,n} = 3\pi n - \dfrac{3\pi}{2} + \dfrac{\alpha_2}{n} + O\left(\dfrac{1}{n^2}\right), \end{cases}$$

where

$$\alpha_1 = \frac{3 + 2mq_1 + 2mq_2}{3\pi m},\ \alpha_2 = -\frac{1 + mq_2}{3\pi m},\ q_1 = q_1\left(\frac{1}{3}\right),\ q_2 = q_2\left(\frac{1}{3}\right).$$

.

Theorem 16. *The eigenfunctions* $y_{i,n}(x)$ *of the problem (4)-(5) corresponding to eigenvalues* $\lambda_{i,n} = (\rho_{1,n})^2, i = 1,2,3;\ n \in N,$ *satisfies the following asymptotic equalities:*

$$y_{1,n}(x) = \begin{cases} \sin 3\pi n x + O\left(\dfrac{1}{n}\right), & x \in \left[0, \dfrac{1}{3}\right], \\ \gamma_{1,n} \sin 3\pi n x + O\left(\dfrac{1}{n}\right), & x \in \left[\dfrac{1}{3}, 1\right], \end{cases}$$

$$y_{2,n}(x) = \begin{cases} \sin 3\pi n x + O\left(\dfrac{1}{n}\right), & x \in \left[0, \dfrac{1}{3}\right], \\ \gamma_{2,n} \sin 3\pi n x + O\left(\dfrac{1}{n}\right), & x \in \left[\dfrac{1}{3}, 1\right], \end{cases}$$

$$y_{3,n}(x) = \begin{cases} O\left(\dfrac{1}{n}\right), & x \in \left[0, \dfrac{1}{3}\right], \\ \gamma_{3,n} \cos 3\pi\left(n - \dfrac{1}{2}\right)x + O\left(\dfrac{1}{n}\right), & x \in \left[\dfrac{1}{3}, 1\right], \end{cases}$$

where $\gamma_{1,n} = (1 + mq_1) + O\left(\frac{1}{n}\right)$, $\gamma_{2,n} = \frac{mq_1 - mq_2}{3} + O\left(\frac{1}{n}\right)$,

$\gamma_{3,n} = m + O\left(\frac{1}{n}\right)$.

Theorem 17. *The root vector system of operator L* $\{\hat{y}_{in}\}_{i=\overline{1,3};n\in N}^{\infty}$ *is complete in space* $L_p(0,1) \oplus C$, $1 < p < \infty$.

Corollary 2. *The root vector system of operator L* $\{\hat{y}_{in}\}_{i=\overline{1,3};n\in N}^{\infty}$ *is complete and minimal in space* $L_p(0,1) \oplus C, 1 < p < \infty$.

Theorem 18. *The necessary and sufficient condition for the system obtained by eliminating any function* $y_{in}(x)$ *from the system of eigenfunctions of the problem (4),(5) be complete and minimal* $L_p(0,1)$ *is that its corresponding function from the biorthogonal system* $z_{in}(x)$ *satisfies the condition* $z_{in}\left(\frac{1}{3}\right) \neq 0$; *here the corresponding biorthogonal function is the first component of the vector* $\hat{z}_{in} = \left(z_{in}(x), \overline{m} z_{in}\left(\frac{1}{3}\right)\right)$ *satisfying the condition* $\langle \hat{y}_{in}, \hat{z}_{in} \rangle = 1$.

Theorem 19. *The root vector system* $\{\hat{y}_{in}\}_{i=\overline{1,3};n\in N}^{\infty}$ *of the operator L forms basis for space* $L_p(0,1) \oplus C$, $1 < p < \infty$, *and if* $p = 2$, *then the root vector system forms a Riesz basis.*

Theorem 20. *The necessary and sufficient condition for the system obtained by eliminating any function* $y_{i,n_0}(x)$ *from the system of eigenfunctions and associated functions* $\{y_0\} \cup \{y_{i,n}\}_{i=1,2;\, n\in N}^{\infty}$ *of the problem (4)-(5) to form a basis for space* $L_p(0,1), 1 < p < \infty$ *and to form a Riesz basis where* $p = 2$, *is that* $z_{i,n_0}(x)$ *which its corresponding function from the biorthogonal system satisfies the* *condition* $z_{i,n_0}\left(\frac{1}{3}\right)\neq 0$. *If* $z_{i,n_0}\left(\frac{1}{3}\right)=0$, *then after eliminating its corresponding* $y_{1,n_0}(x)$ *function, obtaining system does not form a basis for* $L_p(0,1)$, *also the obtaining system will be not complete and minimal in this space.*

The third chapter devoted studying basicity of eigenfunctions of spectral problems in Morrey type spaces.

In section 3.1 was given information about Morrey type spaces. By the $L^{p,\alpha}(-\pi,\pi)$, $1\le p<+\infty$, $0\le\alpha\le 1$ we mean a normed space of all measurable functions on $(-\pi,\pi)$

$$\|f\|_{L^{p,\alpha}(-\pi,\pi)}=\sup_{I\subset[-\pi,\pi]}\left(|I|^{\alpha-1}\int_I|f(t)|^p|dt|\right)^{1/p}<+\infty,$$

Let us consider the subspace $M^{p,\alpha}$, which consists of above continuous functions $f(\cdot)$ in $L^{p,\alpha}$. $\|f(\cdot+\delta)-f(\cdot)\|_{p,\alpha}\to 0,\ \delta\to 0.$ $M^{p,\alpha}$, $1\le p<+\infty$, $0<\alpha\le 1$ is a Banach space and $C_0^\infty[-\pi,\pi]$ is dense here.

In section 3.2 considered the following spectral problem in [-1,1]

$$-y''(x)=\lambda y(x),\quad x\in(-1,0)\cup(0,1),$$

$$y(-1)=y(1)=0,$$
$$y(-0)=y(+0),$$
$$y'(-0)-y'(+0)=\lambda m y(0),$$

There was shown that the abstract method suggested in the work B.T.Bilalov və T.B.Gasymov[7] can be applied also in the non-standart spaces.

Statement 3. *Each of the trigonometric systems* $\{\sin nx\}_{n=1}^{\infty}$ *and* $\{\cos nx\}_{n=0}^{\infty}$ *forms a basis for* $M^{p,\alpha}, 1 < p < +\infty, 0 < \alpha \le 1$.
Denote

$$e_{1,n}(x) = \sin \pi nx, \; x \in [-1,1]$$

$$e_{2,n}(x) = \begin{cases} \sin \pi nx, & x \in [-1,0], \\ -\sin \pi nx, & x \in [0,1], \end{cases}$$

and consider the system $\{\hat{e}_n\}_{n=0}^{\infty}$, *where*

$$\hat{e}_0 = (0;1), \; \hat{e}_{2n} = (e_{2,n};0), \; \hat{e}_{2n-1} = (e_{1,n};0), \; n \in N$$

Theorem 21. *The system* $\{\hat{u}_n\}_{n=0}^{\infty}$ *forms a basis equivalent to the system* $\{\hat{e}_n\}_{n=0}^{\infty}$ *for space* $M^{p,\alpha}(-1,1) \oplus C, 1 < p < \infty, \quad 0 < \alpha \le 1.$

Let us consider the basicity of the system $\{u_n(x)\}_{n=0}^{\infty}$ with eliminating any function form this system in $M^{p,\alpha}(-1,1)$.

Theorem 22. *If* n_0 *-is an arbitrary even number, the system* $\{u_n(x)\}_{n=0, n\ne n_0}^{\infty}$ *forms an equivalent basis to the system* $\{e_n(x)\}_{n=1}^{\infty}$ *for* $M^{p,\alpha}(-1,1), 1 < p < \infty, 0 < \alpha \le 1$. *If* n_0 *- is an arbitrary odd number then the system* $\{u_n(x)\}_{n=0, n\ne n_0}^{\infty}$ *does not form a basis for* $M^{p,\alpha}(-1,1)$, *moreover, it is not complete and minimal in this space.*

In section 3.3 was considered the basis properties of eigenfunctions of spectral problem (9)-(10) in Morrey type spaces.

Theorem 23. *The system of eigenfunctions and associated functions of the operator* L *forms a basis for space* $M^{p,\alpha}(0,1) \oplus C$, $1 < p < \infty, 0 < \alpha \le 1$.

Theorem 24. *If from the system of eigen and associated functions of problem (9)-(10)* $\{y_0\} \cup \{y_{i,n}\}_{i=1,2;\, n\in N}^{\infty}$ *we eliminate any function* $y_{2,n_0}(x)$, *then the obtaining system forms a basis for*

$M^{p,\alpha}(0,1)$, $1 < p < \infty$, $0 < \alpha \leq 1$. And we eliminate any function $y_{1,n_0}(x)$ from this system, then the obtaining system does not form a basis in $M^{p,\alpha}(0,1)$, moreover, in this case the obtained system is not complete and is not minimal in this space.

CONCLUSIONS

The dissertation work is devoted to studying thr properties of basicity of the eigenfunctions and associated functions of second order discontinuous differential operator with spectral parameter in discontinity condition in Lebesgue and Morrey type spaces.

In the work is obtained the following scientific results:

-based on the corresponding systems of subspaces in the direct sum of Banach spaces the theorems about the completeness, minimality, basicity, also *p*-basicity of systems was proved;

-the asymptotics of the eigenvalues and eigenfunctions the discontinuous second order differential operator with spectral parameter in boundary condition was found;

-the resolvent of the linearizing operator of considering spectral problem was constructed, in $L_p \oplus C, L_p, 1 < p < +\infty$ Lebesgue spaces theorems about the completness and minimality of the system of eigenfunctions and associated functions of linearizing operator was proved;

-theorems about the basicity of eigenfunctions and associated functions of discontinuous second order differential operator in $L_p \oplus C$ and $L_p, 1 < p < +\infty,$ Lebesgue spaces, and theorems about the Riesz basicity in $L_2 \oplus C$ və L_2 spaces was proved;

- theorems about the basicity of eigenfunctions and associated functions of discontinuous second order differential operator in $M^{p,\alpha} \oplus C$ və $M^{p,\alpha}, 1 < p < +\infty, 0 < \alpha \le 1,$ Morrey type spaces was proved.

The main clauses of the dissertation work were published in the following scientific works:

1. Gasymov, T.B., Maharramova, G.V. Properties of eigenvalues and eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Abstracts of International Conference dedicated to 80th anniversary of academician F.G.Maksudov,- Baku: 2010, -p.186-187.
2. Maharramova, G.V. The spectral properties of discontinuous differential operator // BSU Republican scientific conference "Actual problems of mathematics and mechanics", - Baku: - 2012, - p. 138.
3. Maharramova, G.V. The basicity properties of eigenfunctions of a discontinuous differential operator with a spectral parameter in boundary condition // Republican scientific conference "Problems of application of mathematics", - Baku: 2013, - p. 56-59.
4. Gasymov, T.B., Maharramova, G.V. Basicity of eigenfunctions in spaces $L_p(0,1)\oplus C$ for one discontinuous problem with spectral parameter in the boundary conditions // On Actual Problems of Mathematics and Informatics, Abstracts of International conference dedicated to the 90-th anniversary of Haydar Aliyev, -Baku: -May 29-31, -2013,- p. 42.
5. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On basis in the direct sum of spaces // Materials of Republican Conference dedicated to the 100th anniversary of prof. A.Sh. Habibzade - Baku: - 2016, - p. 112-115.
6. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On a method for obtaining a complete and minimal system in the direct sum of Banach spaces // International Workshop on "Non-harmonic Analysis and Differential Operators", -Baku: May 25-27, -2016, -p. 29.
7. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On basicity of eigenfunctions of one discontinuous spectral problem in Morrey type spaces // -India : The Aligarh Bulletin of Mathematics, - 2016. v. 35, № 1-2,- p.119-129.

8. Gasymov, T.B., Maharramova, G.V. On one way of obtaining basis from eigenfunctions of discontinuous differential operators // Modern methods of the theory of boundary value problems, Materials International Conference dedicated to 90th anniversary of V.A.Ilin,- Moscow : 2-6 may,- 2018,- p.124-125.
9. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On basicity of eigenfunctions of one discontinuous spectral problem with spectral parameter in discontinuty conditions // “Differential equations and mixed problems”, International Scientific Conference, - Sterlitamak: - 25- 29 iyun, - 2018, - p. 265-268.
10. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On the basicity of eigenfunctions of a second order differential operator with discontinity point, // International Scientific Conference “ Spectral theory and mixed problems”, - Ufa, Russia: - 1-4 october, - 2018,- p.59-61.
11. Maharramova, G.V., On the completeness of eigenfunctions of a second order differential operator // - Baku: Journal of Contemporary Applied Mathematics, - 2018. v.8, №2, - p.45-55.
12. Gasymov, T.B, Maharramova, G.V., Mammadova, N.G. Spectral properties of a problem of vibrations of a loaded string in Lebesgue spaces//-Baku: Transactions of National Academy Sciences of Azerbaijan, Issue Mathematics, Series of Physical-Technical and Mathematical Sciences,-2018. v.38(1),-p.62-68 .
13. Gasymov, T.B, Maharramova, G.V., Jabrailova, A.N. Spectral properties of the problem of vibration of a loaded string in Morrey type spaces//-Baku: Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan,-2018.v. 44(1),-,p.116-122.
14. Gasymov, T.B., Maharramova, G.V. The stability of the basis properties of multiple systems in a Banach space with respect to certain transformations // -Baku: Caspian Journal of Applied Mathematics, Ecology and Economics,- December,-2018.v.6, No 2, p. 66-77.
15. Maharramova G.V. Properties of Eigenvalues and Eigenfunctions of a Spectral Problem with Discontinuity Point //-Baku: Caspian

Journal of Applied Mathematics, Ecology and Economics, - July,- 2019. v.7, No1,-pp.114-125.

16. Bilalov, B.T., Gasymov, T.B., Maharramova, G.V. On basicity of functions of one spectral problem with discontinity point in Lebesgue spaces. //- Moscow : Differential equations,- 2019. T.55, № 12, -p. 1-10.

17. Gasymov, T.B., Maharramova, G.V., Omarova, G.S. The properties of eigenfunctions of one spectral problem with discontinity point // "Mathematics, Mechanics and their applications" Abstracts of International conference dedicated to the 97-th anniversary of Haydar Aliyev ", - Baku:- 20-21 may, - 2020, - p.76-78.

18. Gasymov, T.B., Maharramova, G.V., Kasimov, T.F. Completeness and minimality of eigenfunctions of a spectral problem in spaces $L_p \oplus C$ and L_p - Baku: Journal of Contemporary Applied Mathematics, - 2020. v.10, №2, - p.85-100.

The defense will be held on 17 September 2021 at 14^{00} at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan.

Address: AZ 1141, Baku, B.Vahabzadeh, 9.

Dissertation is accessible at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan Library

Electronic versions of dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan.

Abstract was sent to the required addresses on 02 July 2021.

Signed for print: 29.06.2021
Paper format: 60x84 1/16
Volume: 40000
Number of hard copies: 20