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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

GLOBAL BIFURCATION OF SOLUTIONS FROM INFINITY OF SOME NONLINEAR EIGENVALUE PROBLEMS

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and development degree. Theory of bifurcation of nonlinear eigenvalue problems is one of the important sections of nonlinear analysis. Application of bifurcations of nonlinear eigenvalue problems is very important since such problems appear almost in all fields of natural science: in theory of vibrations, heat convection, in hydrodynamics, in theory of critical modes of atomic and chemical reactors, critical loads. In this direction fundamental results were obtained for a wide class of nonlinear eigenvalue problems. M.A.Krasnoselski[1] and P.H.Rabinowitz[2] have developed local and global theory of bifurcation of solutions of nonlinear eigenvalue problems with completely continuous operators Frescshe differentiable at zero and infinity.

The results of P.H.Rabinowitz[2] were extended on nonlinear eigenvalue problems without assumption of differentiability of nonlinear perturbations in zero by A.P.Mahmudov and Z.S.Aliyev[3]. Bifurcation of solutions from zero of nonlinear eigenvalue problems for the Sturm-Liouville equation was earlier considered by P.H.Rabinowitz, G.Berestycki[4] and B.P.Rynne[5].

These authors have proved the existence of two families of unbounded continua of solutions in $R \times C^1$, possessing ordinary nodal

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[1] Красносельский, М.А. Топологические методы в теории нелинейных интегральных уравнений/М.А.Красносельский.–Москва–Ленинград: Гостех-издат, – 1956. – 392 с

[2] Rabinowitz, P.H. Some global results for nonlinear eigenvalue problems // Journal of Functional Analysis, – 1971. v.7, №3, – p. 487–513.

[3] Махмудов А.П., Алиев З.С. Некоторые глобальные результаты для нелинейных спектральных задач Штурма-Лиувилля четвертого порядка // – Москва: Дифференциальные уравнения, –1993. т. 29, № 8, – с. 1330–1339

[4] Berestycki, H. On some nonlinear Sturm-Liouville problems//Journal of Differential Equations, – 1977. v. 26, №3, – p. 375–390.

[5] Rynne, B.P. Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable // Journal of Mathematical Analysis and Applications. – 1998. v. 228, №1, – p. 141–156

properties and branching from the points and segments of the line of trivial solutions containing eigenvalues corresponding to the linear problem. Global bifurcation of solutions from infinity of nonlinear Sturm-Liuville eigenvalue problems was studied by P.H.Rabinowitz[6] and B.P.Rynne[5].

For these problems the authors show the existence of two families of unbounded continua of solutions, branching from the points and segments $R\times\{\infty\}$ and possessing ordinary nodal properties in the vicinity of these points and segments.

Nonlinear eigenvalues problems for fourth order ordinary differential equations were considered in many papers in different statements the great majority of which were devoted to the study of bifurcation of solutions in the classes of positive and negative functions. Note that in nonlinear eigenvalue problems for fourth order ordinary differential equations the nodal properties are not preserved along the continua of solutions and therefore it is impossible to investigate the structure of behavior of global continua of solutions in detail using the above mentoned works of P.G.Rabinovich, H.Berestycki and B.P.Rynne. In his recent paper [7] Z.S.Aliyev[7] has completely studied global bifurcation from zero of solutions of nonlinear eigenvalue problems for fourth order ordinary differential equations with completely boundary conditions. In this paper, to retain the nodal properties, using Prufer type transform the classes of functions in $R\times C^3$, that posses oscillatory properties of eigenfunctions and their derivatives corresponding to the linear problem were constructed and the existence of two families of unbounded continua of solutions of the considered problems contained in these sets and branching from the points and straight line of trivial

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[6] Rabinowitz, P.H. On bifurcation from infinity//Journal of differential equations – 1973. v.14, no. 3, – p. 462–475

[7] Алиев, З.С. О глобальной бифуркации решений некоторых нелинейных задач на собственные значения для обыкновенных дифференциальных уравнений четвертого порядка // – Москва: Математический сборник, – 2016. т. 207, № 12, – с. 3–29

solutions, was proved. From the above it follows that global bifurcation of solutions from the infinity of nonlinear eigenvalue problems for fourth order ordinary differential equations was not actually studied. Consequently, the study of the structure of asymptotic bifurcation points, structure and behavior of unbounded continua of solutions of fourth order nonlinear eigenvalue problems branching from the points and segments of $R\times\{\infty\}$ is an urgent problem.

Goal and tasks of the study. The main goal and task of the work is to study global bifurcation of solutions from infinity of nonlinear nonlinearized eigenvalue problems in Banach and Hilbert spaces; to study the behavior of continua of solutions branching from $R\times\{\infty\}$, nonlinear eigenvalue problems for fourth order ordinary differential equations with completely regular boundary conditions..

İnvestigation methods. In the work, the methods of functions theory and functional analysis, theory of operators, theory of differential equations, of spectral analysis, spectral theory of differential operators, topology, nonlinear analysis and bifurcation theory are used.

Basic statements to be defended. The following basic statements are defended:

– to define segments of asymptotic bifurcation and to study global continua of solutions of nonlinearized eigenvalue problems considered in Banach and Hilbert spaces branching from asymptotic segments of bifurcation;

– to study global bifurcation of solutions from infinity of linearized eigenvalue problems for completely regular Sturm systems of fourth order and on the specific examples to show all possible cases of behavior of global continua of solutions;

– to determine the segments of bifurcation according to the certain oscillatory class and to study the behavior of global continua of solutions of nonlinearized eigenvalue problems for fourth order ordinary differential equations.

Scientific novelty of the study. In the dissertation work the following main results were obtained:

– the structure of asymptotic bifurcation points was studied, the behavior of unbounded connected component coming from infinity, the solutions of some nonlinear (nonlinearized) eigenvalue problems was studied;

– behavior of structure of continua of solutions branching both from infinity and from zero and from infinity, of linearized eigenvalue problems for fourth order ordinary differential equation with completely regular boundary conditions was studied;

– the structure of asymptotic bifurcation points was studied, behavior and structure of connected components of solutions branching from asymptotic bifurcation segments of nonlinearized eigenvalue problems for fourth order ordinary differential equations with completely regular boundary conditions, were researched.

Theoretical and applied value. The results obtained in the work are of theoretical character. They may be used in various issues of theory of differential equations, of nonlinear analysis, bifurcation theory and when studying various problems of mechanics and physics.

Approbation and application. The result of dissertation work were reported at the seminars of the chair of "Mathematical analysis" of Ganja State University (the head ass. prof. A.M.Huseynov), of the chair of “Mathematical analysis" of Baku State University (the head prof. S.S.Mirzoyev), at the seminar of the department of "Differential equations" (the head: prof. A.B.Aliyev) of IMM, at the international conference "Mathematical analysis, differential equations and their applications" (MADEA-7, Baku, 2015), at the international conference «Theoretical and applied problems of mathemtics» dedicated to 55 years of Sumgayit State University (Sumgayit, 2017), at the international (51-th all Russian) youth school conference “Contemporary problems of mathematics and its applications” (Ekaterinburgh, Russia, 2020) .

Author’s personal contribution is in formulation of research goal. Furthermore, all the results of the study belong to the author.

Author's publications. Publications in editions recommended by Higher Attestation Commission under President of the Republic of Azerbaijan – 5, conference materials – 2, abstracts of reports – 1.

The institution where the dissertation work was performed. The work was performed at the chair of "Mathematical analysis" of Ganja State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately). Total volume of the dissertation work– 203998 signs (title page – 405 signs, contents – 2460 signs, introduction – 48000 signs, chapter I – 62000 signs, chapter II – 46000 signs, chapter III – 44000 signs, conclusion – 1133 signs). The list of references consists of 83 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, three chapters, conclusions and a list of references.

In the introduction we justify the rationale of the studied is topic and show the degree of its development, formulate the goal and tasks of the study, give scientific novelty, note theoretical and practical value of the study and give information on approbation of the work.

In chapter I we study bifurcation of solutions of nonlinear eigenvalue problems in the Hilbert space without assumption of differentiability at the infinity of nonlinear perturbations. In this chapter the above mentioned results of M.A.Krasnoselski and P.H.Rabinowitz are extended to these nonlinear eigenvalue problems. The main results of this chapter may be found in the author's papers [1, 2, 4].

In 1.1 we give necessary information from theory of bifurcation of nonlinear eigenvalue problems.

In 1.2 we study global bifurcation from zero of nonlinearized eigenvalue problems.

In 1.3 we study global bifurcation from infinity of nonlinearized eigenvalue problems.

Let H be a real Hilbert space with the norm $\|\cdot\|$, $L: D(L) \subset H \to H$ ($D(L)$ is every where dense in H) be a linear, lower-bounded self-adjoint operator with a compact resolvent. Note that each eigenvalue of the operator L is real, isolated and has a finite multiplicity, and the spectrum $\sigma(L)$ consists only of infinitely non-decreasing sequence of such points.

Let us consider the following nonlinear eigenvalue problem

$$Lu = \lambda u + \tilde{F}(\lambda, u) + \tilde{G}(\lambda, u), \tag{1}$$

where the operators $\tilde{F}: R \times H \to H$ и $\tilde{G}: R \times H \to H$ are continuous and satisfy the following conditions: there exists such a number $M > 0$ that

$$\|\tilde{F}(\lambda, u)\| \le M \|u\| \text{ for } \lambda \in R \text{ and } \|u\| < 1; \tag{2}$$

$$\tilde{G}(\lambda, u) = o(\|u\|) \text{ for } \|u\| \to 0, \tag{3}$$

uniformly with respect to $\lambda \in \Lambda$ for each bounded interval $\Lambda \subset R$.

We define the norm in the space $R \times H$ as follows:

$$\|(\lambda, u)\| = \left\{ |\lambda|^2 + \|u\|^2 \right\}^{\frac{1}{2}}.$$

By $B_r(\lambda)$ we denote a sphere in $R \times H$ centered at the point $(\lambda, 0)$ and of radius r, by B_r a sphere in H centered at the point 0 and of radius r.

They say that (μ, ∞) is a bifurcation point or asymptotic bifurcation point of problem (1) if there exists the sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset R \times H$ of solutions of this problem such that $\lambda_n \to \mu$ and $\|u_n\| \to +\infty$ as $n \to \infty$.

Let $I_\mu = [\mu - M, \mu + M]$. Denote by B^∞ the set of asymptotic bifurcation points of problem (1).

Theorem 1. Let $\mu \in \sigma(L)$ have an odd multiplicity and conditions (2), (3) and

$$dist\{\mu, \sigma(L) \setminus \{\mu\}\} > 2M. \tag{4}$$

be fulfilled. Then $B^\infty \cap (I_\mu \times \{\infty\}) \ne \varnothing$.

Lemma 1. If $(\lambda, \infty) \in B^{\infty}$, then $dist\{\lambda, \sigma(L)\} < M$.

Corollary 1. If μ is an eigenvalue of the operator L of odd multiplicity and condition (4) is fulfilled, then $B^{\infty} \bigcap ((I_{\mu}(\delta_{\mu,0}) \setminus I_{\mu}) \times \{\infty\}) = \varnothing$ where $I_{\mu}(\delta_{\mu,0}) = [\mu - M - \delta_{\mu,0}, \mu + M + \delta_{\mu,0}]$, $\delta_{\mu,0} = (dist\{\mu; \sigma(L) \setminus \{\mu\}\} - 2M)/2$.

Let $\Im$ be a set of nontrivial solutions of problem (1) in $R \times H$. Define the set $\Im_{\mu}^{*}$ as the union of all components $\Im_{\mu,\lambda}^{*}$ of the set $\Im$ branching from asymptotic bifurcations points $(\lambda, \infty) \in B^{\infty} \bigcap (I_{\mu} \times \{\infty\})$. Note that the set $\Im_{\mu} = \Im_{\mu}^{*} \bigcup (I_{\mu} \times \{\infty\})$ is connected in $R \times E$.

For each set $A \subset R \times E$ by $P_R(A)$ we denote the projection of the set A on to R, by $P_E(A)$ the projection of the set A on to E.

From theorem 1 it follows that the set $\Im_{\mu}$ contains $(I_{\mu} \times \{\infty\})$ and is unbounded in $R \times E$. Furthermore, we have the following theorem

Theorem 2. Suppose that $\mu \in \sigma(L)$ has an odd multiplicity, the condition (4) is fulfilled, the operators

$$(\lambda, u) \to \| u \|^2 \, F\left(\lambda, \frac{u}{\| u \|^2}\right) \quad \text{and} \quad (\lambda, u) \to \| u \|^2 \, G\left(\lambda, \frac{u}{\| u \|^2}\right)$$

are L compact. Furthermore, let $\Lambda \subset R$ be a such a segment in R that $I_{\mu} \subset \Lambda \subset I_{\mu}(\delta_{\mu,0})$ and C_{μ} be such a vicinity of the segment $I_{\mu} \times \{0\}$ in $R \times E$ that $P_R(C_{\mu}) \subset \Lambda$ and $dist\{0, P_E(C_{\mu})\} > 0$. Then either (i) $\Im_{\mu} \setminus C_{\mu}$ is bounded in $R \times E$, and in this case $(\Im_{\mu} \setminus C_{\mu}) \bigcap \{(\lambda, 0) : \lambda \in R\} \neq \varnothing$, or (ii) $\Im_{\mu} \setminus C_{\mu}$ is unbounded in $R \times E$, and if $P_R(\Im_{\mu} \setminus C_{\mu})$ is bounded in R, then $\Im_{\mu}$ contains another segment $I_{\tilde{\mu}} \times \{\infty\}$, where $\mu \neq \tilde{\mu} \in \sigma(L)$.

Now let $\mu \in \sigma(L)$ be a simple and $\vartheta \in D(L)$ be an appropriate normalized eigen vector. Then the following holds

Theorem 3. $\Im_\mu$ can be expanded into two subcontinua $\Im_\mu^+$, $\Im_\mu^-$ and there exists such a vicinity $Q_\mu \subset C_\mu$ of the segment $I_\mu \times \{\infty\}$ that if $(\lambda, u) \in \Im_\mu^+ (\Im_\mu^+) \cap D_\mu$ and $(\lambda, u) \in (\mu, \infty)$, then $(\lambda, u) \in (\lambda, \alpha\vartheta + w)$, where $\alpha > 0$ $(\alpha < 0)$ and $|\lambda - \mu| = M + o(1)$, $w = o(|\alpha|)$ as $|\alpha| \to \infty$.

In this section we give an application to the nonlinear eigenvalue problem for second order ordinary differential equations.

In chapter II we consider a linearized at infinity, i.e. asymptotically linear eigenvalue problem for fourth order ordinary differential equations with completely regular boundary conditions and prove the existence of two families of unbounded continua of solutions branching from asymptotic bifurcation points and possessing ordinary nodal properties near these points. The main results of this chapter were published in the author's papers [5, 7, 8].

In 2.1 we give the problem statements and some historical remarks.

We consider the equation.

$$\begin{aligned} \ell(y) \equiv (py'')'' - (qy')' + r(x)y = \\ = \lambda \tau y + g(x, y, y', y'', y''', \lambda),\ x \in (0, l), \end{aligned} \tag{5}$$

under boundary conditions

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0,$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{6a}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0,$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{6б}$$

where $\lambda \in R$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, the functions $p \in C^2[0,l],\ q \in C^1[0,l],\ r, \tau \in C[0,l],\ p(x) > 0, q(x) \geq 0,\ \tau(x) > 0$ on $[0, l]$, $\alpha, \beta, \gamma, \delta$ – are real constants such that $0 \leq \alpha, \beta, \gamma, \delta \leq \pi/2$, except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ и

$\alpha = \gamma = \beta = \delta = \pi/2.$ The nonlinear term g is continuous on $[0, l] \times R^5$ and we will assume that either the first or both of the following conditions is fulfilled for it: for each bounded interval $\Lambda \subset R.$

$$g(x,u,s,v,w,\lambda) = 0\,(|u|+|s|+|v|+|w|) \quad \text{as} \quad |u|+|s|+|v|+|w| \to \infty, \tag{7}$$

$$g(x,u,s,v,w,\lambda) = 0(|u|+|s|+|v|+|w|) \quad \text{as} \quad |u|+|s|+|v|+|w| \to 0, \tag{8}$$

uniformly with respect to $x \in [0,l]$ and $\lambda \in \Lambda$.

If the nonlinear term g satisfies only the condition (8), then we consider bifurcation from the point $y = 0.$

Note that A.C.Lazer and P.J.McKenna[8], R.Ma.and B.Tompson[9] have obtained the results similar to the results of the paper of P.H.Rabinowitz for the problem (5)-(6) in the case when $q \equiv 0,$ $\alpha = \gamma = \pi/2,$ $\beta = \delta = 0,$ and the nonlinear term g satisfies the smallness condition for $y = 0$ of the form $g(y) = o(|y|),$ that enable to retain nodal properties of solutions.

In Z.S.Aliyev's paper local and global bifurcation of solutions of the problem (5)-(6) subject to the condition (8), was completely studied in the case when $r \equiv 0.$ For studying bifurcation of solutions of problem (5)-(6) subject to the condition (8) in this work using the Prufer type transformations the classes of functions that retain nodal properties of eigen functions and their derivatives of the linear problem obtained from (5)-(6) for $r \equiv 0$ and $g \equiv 0$ are constracted Furthermore, the existence of two families of unbounded continua of solutions contained in these classes was proved.

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[8] Lazer, A.C., McKenna, P.J. Global bifurcation and a theorem of Tarantello // Journal of Mathematical Analysis and Applications, – 1994. v.181, no. 3, – p. 648–655

[9] Ma, R., Tompson, B. Nodal solutions for a nonlinear fourth order eigenvalue problem // Acta Mathematica Sinica, English Series, – 2008. v.24, no.1, – p. 27–34

If only the condition (7) is fulfilled, then bifurcation from $y=\infty$ is considered. A similar problem for the Sturm-Liouville equation was studied by H.Rabinowitz.

It should be noted that only I.Przybycin[10] has obtained the results similar to the results of P.H.Rabinowitz for a special class of nonlinear eigenvalue problems for fourth order ordinary differential equations in the case when the nonlinear term g satisfies the smallness condition of the form $g(x,y,s,\vartheta,\lambda)=o(|y|)$ as $y\to\infty$.

The goal of this chapter is to study bifurcation of solutions of nonlinear problem (5)-(6) in the following cases: (i) only condition (7); (ii) both conditions (7) and (8) are fulfilled.

In 2.2 we give some auxiliary statements. Let $E=C^3[0,l]\bigcap B.C.$ be a Banach space with the ordinary norm

$$\|u\|_3=\|u\|_\infty+\|u'\|_\infty+\|u''\|_\infty+\|u'''\|_\infty,$$

$\|u\|_\infty=\max\limits_{x\in[0,\,l]}|u(x)|$.

Following Z.S.Aliyev's for each $k\in\mathrm{N}$ and for each $\nu\in\{+,-\}$, by S_k^ν we denote the set of functions from E possessing oscillatory properties of eigenfunctions and their derivatives of the linear problem obtained from (5)-(6) for $r\equiv 0$ и $g\equiv 0$.

By virtue of theorem 1.2 of Z.S.Aliyev's above mentioned paper the eigenvalues of the linear problem

$$\begin{cases}(py'')''-(qy')'+r(x)y=\lambda\tau y,\ x\in(0,l),\\ y\in B.C.,\end{cases} \tag{9}$$

are real, simple and form unboundedly increasing sequence $\{\lambda_k\}_{k=1}^\infty$, where $B.C.$ denotes the set of functions satisfying the boundary conditions (6); for each $k\in\mathrm{N}$ the eigen function $y_k(x)$, corresponding to the eigenvalue λ_k, is contained in S_k (consequently,

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[10] Przybycin J. Some applications of bifurcation theory to ordinary differential equations of the fourth order // Annales Polonici Mathematici, – 1991. v. 53, no. 2, – p. 153–160.

$y_k(x)$ has precisely $k-1$ simple nodal zeros in $(0, l)$). By C we denote the set of solutions of problem (5)-(6). We say that the set $D \subset C$ intersects the point (λ, ∞), $\lambda \in R$, (the point $(\lambda, 0)$ respectively), if there exists such a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ that $\lambda_n \to \lambda$ and $\| u_n \|_3 \to \infty$ (respectively, $\| u_n \|_3 \to 0$). Furthermore, we say that $D \subset C$ intersects (λ, ∞) (respectively, $(\lambda, 0)$) along the set $R \times S_k^{\nu}$, $k \in \mathrm{N}$ and $\nu \in \{+, -\}$, if the sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset D$ can be chosen so that $u_n \in S_k^{\nu}$ for all $n \in \mathrm{N}$ (in this case we say that (λ, ∞) (respectively, $(\lambda, 0)$) is a bifurcation point of the problem (8)-(9) over the set $R \times S_k^{\nu}$).

Theorem 4. Let the condition (8) be fulfilled. Then for each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ there exists the continuum $\mathfrak{I}_k^{\nu}$ of solutions of problem (5)-(6) in $(R \times S_k^{\nu}) \cup \{(\lambda_k, 0)\}$, that intersects $(\lambda_k, 0)$ and (λ_k, ∞) in $R \times E$.

In 2.3 we study asymptotic bifurcation of solutions of problem (5)-(6), where it is assumed that only condition (7) is fulfilled.

One of the main results of this chapter is the following theorem

Theorem 5. For each $k \in \mathrm{N}$ and each $\nu \in \{+, -\}$ there exists a connected component C_k^{ν} of the set C, that contains (λ_k, ∞) and possesses the following properties: (i) there exists such a vicinity Q_k of the point (λ_k, ∞) in $R \times E$ that $Q_k \cap (C_k^{\nu} \setminus \{(\lambda_k, \infty)\} \subset R \times S_k^{\nu}$; (ii) either C_k^{ν} intersects $C_{k'}^{\nu'}$ along the set $R \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$, or C_k^{ν} intersects $(\lambda, 0)$ for some $\lambda \in R$, or $P_R(C_k^{\nu})$ is unbounded in R.

Remark 1. It should be noted that in theorem 5 it is not affirmed that $C_k^{\nu} \subset (R \times S_k^{\nu}) \cup \{(\lambda_k, \infty)\}$.

Remark 2. If for each $\lambda \in R$ there exists such a $x \in (0, l)$ that $g(x,0,0,0,0,\lambda) \neq 0,$ then the second statement in (ii) of theorem 5 does not hold.

If we impose additional conditions on the function $g,$ then we can more exactly describe the set of solutions of problem (5)-(6).

Corollary 2. Let the function g be representable in the form

$$g(x,u,s,\vartheta,w,\lambda) = g_1(x,u,s,\vartheta,w,\lambda)u + g_2(x,u,s,\vartheta,w,\lambda)s +$$
$$g_3(x,u,s,\vartheta,w,\lambda)\vartheta + g_4(x,u,s,\vartheta,w,\lambda)w,$$

where $g_1,\ g_2, g_3$ and g_4 are the functions continuous at the point $(u,s,\vartheta,w) = (0,0,0,0)$. Then $C_k^\nu \setminus Q_k$ contains a subcontinimum that lies in $R \times S_k^\nu$ and either is unbounded in $R \times E,$ or intersects the set $\Re = \{(\lambda,0) : \lambda \in R\}.$

Corollary 3. If g is the same as in corollary 2, and $g_i(x,0,0,0,0,\lambda) = 0,\ i = 1,2,3,4,$ and C_k^ν intersects $\Re,$ then the intersection point of the set C_k^ν with the set $\Re = \{(\lambda,0) : \lambda \in R\}$ is $(\lambda_k, 0).$

In 2.4 we consider bifurcation of solutions of problem (5)-(6) if the conditions (11) and (10) are fulfilled simultaneously. Here we can improve the statements of theorems 4 and 5 as follows:

Theorem 6. If both conditions (7) and (8) are fulfilled, then for each $k \in \mathrm{N}$ and $\nu \in \{+,-\}$ the relation $C_k^\nu \subset R \times S_k^\nu$ is valid and the statement (i) of theorem 5 does not hold. Moreover, if $\Im_k^\nu$ intersects the point $(\lambda, \infty),$ for some $\lambda \in R,$ then $\lambda = \lambda_k.$ In the same way, if C_k^ν intersects $(\lambda, 0)$ for some $\lambda \in R,$ then $\lambda = \lambda_k.$

Naturally, there arises a question if the sets $\Im_k^\nu$ and C_k^ν intersect. In this section we give examples that shows that both cases are possible.

In chapter III we study global bifurcation from infinity of solutions of a nonlinear eigenvalue problem (5)-(6) in the case when the nonlinear term is not differentiable at infinity. The basic results of this chapter have been published in the author's papers [3, 5, 6].

In 3.1 we give the problem statement and historical remarks.

We consider a nonlinear eigenvalue problem for the equation

$$\begin{gathered}\ell(y) \equiv (py'')'' - (qy')' + r(x)y = \\ = \lambda \tau y + h(x, y, y', y'', y''', \lambda), \;\; x \in (0, l),\end{gathered} \tag{10}$$

under boundary conditions (6), where the functions $p(x), q(x)$ $r(x)$, $\tau(x)$ and constants $\alpha, \beta, \gamma, \delta$ satisfy the conditions imposed on them in chapter II.

We represent the nonlinear term h in the form $h = f + g$, where $f, g \in C([0, l] \times R^5)$, g satisfies the condition (7), f satisfies the following condition: there exist such a number $M > 0$ and a rather large number $c_0 > 0$ that

$$\begin{gathered}| f(x, u, s, \vartheta, w, \lambda)/y | \le M, x \in [0, l], u, s, \vartheta, w \in R, \\ |u| + |s| + |v| + |w| \ge c_0, \; \lambda \in R.\end{gathered} \tag{11}$$

If g satisfies (8), f satisfies the condition

$$\begin{gathered}| f(x, u, s, \vartheta, w, \lambda)/y | \le M, x \in [0, l], u, s, \vartheta, w \in R, \\ |u| + |s| + |v| + |w| \le 1, \; \lambda \in R,\end{gathered} \tag{12}$$

uniformly with respect to $x \in [0, l]$ и $\lambda \in \Lambda$, we consider bifurcation from $y = 0$.

In the above mentioned paper Z.S.Aliyev has studied global bifurcation of solutions of problem (10), (6) (in the case $r \equiv 0$) subject to conditions (8) and (12), where he proved the existence of two families of unbounded continua of solutions of this problem branching from the segments of the straightline of trivial solutions and contained in the classes $R \times S_k^{\nu}$.

If conditions (7) and (11), are fulfilled, then the bifurcation from $y = \infty$ is considered. Recall that for the Sturm-Liouville equation, the similar problems were earlier considered in Przybicin's[11] paper and B.P.Rinny's above mentioned paper. For the considered problems, these authors have proved the existence of two families of unbounded continua of solutions branching from the segments $R \times \{\infty\}$ and possessing ordinary nodal properties in the vicinity of these segments.

The goal of this chapter is to study bifurcation from infinity of solutions of problem (10), (6) subject to conditions (7) and (11).

In 3.2 we prove some necessary statements. For studying bifurcations of solutions of nonlinear problem (10), (6) we use the transformation $y \to y / \| y \|_3^2$, that transforms the bifurcation problem from infinity to the appropriate problem of bifurcation from zero. But in this case the set $\{ y \in E : |y| + |y'| + |y''| + |y'''| \geq c_0 \}$ is not transformed into the set of the form $\{ y \in E : |y| + |y'| + |y''| + |y'''| \leq r_0 \}$ for some rather small $r_0 > 0$. Consequently, it is impossible to apply the results of Z.S.Aliyev's paper to the problem of bifurcation from zero. Therefore, for solving this problem we need the following result.

Lemma 2. There exist such functions f^* and g^* that h can be represented in the form $h = f^* + g^*$, where f^* and g^* satisfy the conditions:

$$\left| \frac{f^*(x, u, s, \vartheta, w, \lambda)}{y} \right| \leq M, x \in [0, l],$$

$$(u, s, \vartheta, w, \lambda) \in [0, l] \times R^5, u \neq 0; \tag{13}$$

$$g^*(x, u, s, v, w, \lambda) = 0\left(|u| + |s| + |v| + |w|\right) \quad \text{for} \quad |u| + |s| + |v| + |w| \to \infty, \tag{14}$$

[11] Przybycin, J. Bifurcation from infinity for the special class of nonlinear differential equations // Journal of Differential Equations, – 1986. v. 65, № 2, – p. 235–239

uniformly with respect to $x \in [0,l]$ and $\lambda \in \Lambda$ for every bounded interval $\Lambda \subset R$.

Recall that if 0 is not an eigenvalue of the linear problem (9), then the nonlinear problem (10), (6) is reduced to the equivalent integral equation

$$y(x) = \lambda \int_0^l K(x,t)\tau(t)y(t)dt +$$

$$+\int_0^l K(x,t) f^*(t, y(t)y'(t), y''(t), y'''(t), \lambda)dt +$$

$$+\int_0^l K(x,t) f^*(t, y(t)y'(t), y''(t), y'''(t), \lambda)dt. \tag{15}$$

Let

$$F^*(\lambda, y)(x) = \int_0^l K(x,t) f^*(t, y(t)y'(t), y''(t), y'''(t), \lambda)dt. \tag{16}$$

$$G^*(\lambda, y)(x) = \int_0^l K(x,t) g^*(t, y(t)y'(t), y''(t), y'''(t), \lambda)dt. \tag{17}$$

Obviously, the operator $F^* : R \times E \to E$ is completely continuous, the operator $G^* : R \times E \to E$ is continuous and satisfies the condition

$$G^*(\lambda, y) = o(\| y \|_3) \text{ for } \| y \|_3 \to \infty, \tag{18}$$

uniformly on the bounded $\lambda -$intervals. Furthermore, the operator

$$H : (\lambda, y) \to \| y \|_3^2 \, G^*\left(\lambda, \frac{y}{\| y \|_3^2}\right), \quad H^*(\lambda, 0) = 0,$$

is completely continuous.

By virtue of (15)-(17) we can write problem (10), (6) in the following equivalent form

$$y = \lambda L y + F^*(\lambda, y) + G^*(\lambda, y). \tag{19}$$

In 3.3 we study global bifurcation from infinity of solutions of problem (10), (6).

For studying bifurcation of solutions (10), (6) subject to conditions (7) and (11) we consider the following approximation problem

$$\begin{cases} \ell(y) = \lambda \tau\, y + \dfrac{f^*(x, \| y \|_3^{\varepsilon}\, y, \| y \|_3^{\varepsilon}\, y', \| y \|_3^{\varepsilon}\, y'', \| y \|_3^{\varepsilon}\, y''', \lambda)}{\| y \|_3^{2\varepsilon}} + \\ \qquad + g^*(x, y, y', y'', y''', \lambda),\ x \in (0,\, l), \quad y \in B.C., \end{cases} \tag{20}$$

where $\varepsilon \in (0,\, 1]$.

Lemma 3. Let $\delta > 0$ be a rather small number. Then for each $k \in \mathrm{N}$ there exists such a rather large $R_k^* > 0$ that for each given $\varepsilon \in (0,\, 1)$ the problem (20) does not have nontrivial solution $(\lambda,\ y)$, that satisfies the conditions $dist\{\lambda, I_k\} > \delta,\ y \in S_k^{\nu},\ \nu \in \{+\,,\,-\}$ and $\| y \|_3 > R_k^*$.

Lemma 4. For any rather small $\varepsilon > 0$ there exists such a rather large $\rho_\varepsilon > 0$ that for all $\lambda \in \Lambda$ and $y \in E, \| y \|_3 > \rho_\varepsilon$ we have

$$| g^*(x, y, y', y'', y''', \lambda) | < \varepsilon \| y \|_3,\ x \in [0,\, l].$$

Let $p_0 = \min\limits_{x \in [0,\, l]} p(x)$. We define the numbers $r_k,\ k \in \mathrm{N}$, as follows:

$$r_k = p_0^{-1}\{2 \| p \|_2 + \| q \|_1 + \| r \|_\infty + \\ + (| \lambda_k | + M/\tau_0 + 1) \| \tau \|_\infty + M/R_k^*\}.$$

Lemma 5. Let $\delta > 0$ and $\varepsilon_k,\ k \in \mathrm{N}$ be such a rather small number that $\varepsilon_k < \dfrac{p_0}{2le^{(r_k+1)l}}$. Then for each $\varepsilon_k,\ k \in \mathrm{N}$, there exists such a rather large number $R_k > \max\{R_k^*, \rho_{\varepsilon_k}\}$ that for any $R > R_k$ the problem (10), (6) has the solution $(\lambda_{R,k}^{\nu}, y_{R,k}^{\nu})$ that satisfies the conditions: $dist\{\lambda_{R,k}^{\nu},\ I_k\} \le \delta,\quad \vartheta_{R,k}^{\nu} \in S_k^{\nu},\quad \nu \in \{+\,,\,-\}$ and $\| \vartheta_{R,k}^{\nu} \|_3 = R.$

Corollary 4. The set of asymptotic bifurcation points of problem (10), (6) (or 19) over the set $R\times S_k^\nu$ is non empty. Furthermore, if (λ, ∞) is a bifurcation point for problem (10), (6) over the set $R\times S_k^\nu$, then $\lambda\in I_k$.

By D we denote the set of nontrivial solutions of problem (10), (6).

For each $k\in \mathrm{N}$ and each $\nu\in\{+\,,-\}$ we determine the set $D_k^\nu\subset D$ as a union of all connectivity components of the set D, that intersect $I_k\times\{\infty\}$ along the set $R\times S_k^\nu$.

The main result of this chapter is the following theorem

Theorem 7. For each $k\in \mathrm{N}$ and each $\nu\in\{+,-\}$ for the set D_k^ν at least one the following statements is fulfilled: (i) D_k^ν intersects $I_{k'}\times\{\infty\}$ for some $(k',\nu')\neq(k,\nu)$; (ii) D_k^ν intersects $\Re$ for some $\lambda\in R$; (iii) $P_R(D_k^\nu)$ is unbounded. Furthermore, if the union $D_k=D_k^+\bigcup D_k^-$ does not satisfy (ii) or (iii), then it must satisfy (i) for some $k'\neq k$.

CONCLUSION

In the dissertation work we consider nonlinear eigenvalue problems for fourth order ordinary differential equations with completely regular boundary conditions. We study the structure of asymptotic bifurcation points and the structure of unbounded continua of solutions of the considered problems branching from the points and segments $R\times\{\infty\}$.

– the topic of the study if urgent the considered problems are encountered in mechanics and physics. In the work we obtained the following results.

– the structure of asymptotic bifurcation points was studied, the

behavior of unbounded connected component branching from infinity, of solutions of some nonlinearized eigenvalue problems is researched;
– behavior and structure of continua of solutions branching as from infinity as well as from zero and from infinity, of linearized eigenvalue problems for fourth order ordinary differential equations with completely regular boundary conditions were studied;
– the structure of asymptotic bifurcation points was studied, behavior and structure of connected components of solutions branching from the segments of nonlinearized eigenvalue problems for fourth order ordinary differential equations with completely regular boundary conditions, were investigated.

The basic results of the dissertation work are in the following works:

1. Aliyev, Z.S., Mustafayeva, N.A. Global bifurcation from infinity of some nonlinearizable eigenvalue problems//International conference "Mathematical analysis, differential equations and their applications", MADEA–7, –Baku: – 8–13 September, – 2015, – p. 19.
2. Aliyev, Z.S., Mustafayeva, N.A. Bifurcation from infinity for some nonlinear eigenvalue problems which are not linearizable // – Baku: Transactions of NAS of Azerbaijan, ser. phys.–tech. math. sci., math., – 2015. v. 35, no. 4, – p. 13–18.
3. Мустафаева, Н.А. Глобальная бифуркация решений из бесконечности некоторых нелинейных задач четвертого порядка // "Riyaziyyatın nəzəri və tətbiqi problemləri" Beynəlxalq Elmi Konfransının materialları, – Sumqayıt: – 25–26 may, – 2017, – s.160–161.
4. Алиев З.С., Мустафаева, Н.А. Глобальная бифуркация решений из бесконечности для некоторых нелинейных задач на собственные значения//–Баку: Вестник Бакинского Университета, сер. физ.-мат. наук, – 2018. № 4, – с.1–4.
5. Aliyev, Z. S., Mustafayeva, N.A. Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth order

ordinary differential equations// Electronic Journal of Differential Equations, – 2018. v. 2018, №98, – p. 1–19.
6. Mustafayeva, N.A. On global bifurcation from zero and infinity in fourth order nonlinear eigenvalue problems // Caspian Journal of Applied Mathematics, Ecology and Economics, – 2018. v. 6, № 1, – p. 103–110.
7. Mustafayeva, N.A. Bifurcation from zero and infinity in nonlinear eigenvalue problems for ordinary differential equations of fourth order // Тезисы Международной (51-й Всероссийской) молодёжной школы-конференции “Современные проблемы математики и ее приложений”, – Екатеринбург: – 3–7 февраля, – 2020, – с. 43-44.
8. Mustafayeva, N.A. On global continua of solutions bifurcating from zero and infinity of some nonlinear fourth order eigenvalue problems // Transactions of NAS of Azerbaijan, ser. phys.–tech. math. sci., math., – 2020. v.40, №1, – p. 146-151.

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