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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

PROPERTIES OF THE EIGENFUNCTIONS OF A DIFFERENTIAL EQUATION CORRESPONDING TO THE TRANSVERSE VIBRATIONS OF THE ROD UNDER THE ACTION OF AXIAL AND TRACKING FORCES

Specialty: 1211.01 – Differential Equations

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and development degree. This dissertation is devoted to the study of the spectral properties of ordinary differential operators of fourth order with a spectral parameter in the boundary conditions.

The spectral theory of ordinary differential operators with a spectral parameter in the boundary conditions is one of the important branches of modern mathematics. The study of boundary value problems with a parameter in boundary conditions is of both theoretical and applied interest, since problems of this type are encountered in mechanics and physics. Back 1820, M. Poisson solved the problem of the motion of a body suspended from the end of an inextensible thread, A.N. Krylov and S.P. Timoshenko consider the problem of longitudinal vibrations of a rod as one of the urgent problems of natural science. V.V. Bolotin[1] studies bending vibrations of a rod, in cross-sections of which a longitudinal force acts, the left end of which is fixed, and at the right end an inertial load is concentrated and a tracking force acts. This problem is described by a boundary value problem for ordinary differential equations of the fourth order with a spectral parameter in the boundary conditions.

Sturm-Liouville problems with a spectral parameter in the boundary conditions were previously studied in the works of J. Walter, C.T. Fulton, E.M. Russakovsky, where these problems are reduced to spectral problems for self-adjoint operators in Hilbert spaces $L_2 \oplus C^N, N \geq 1.$ They established that the system of root vectors of these operators forms a Riesz basis in $L_2 \oplus C^N, N \geq 1.$ Later E.I. Moiseev and N.Yu. Kapustin[2] was the first time to establish the basis property in the space $L_p, 1 < p < \infty,$ of systems of root functions with one remote function of the Sturm-Liouville problem with a spectral parameter in the boundary

[1] Vibrations in technology: A reference book in 6 volumes. Vol. 1. Vibrations of mechanical systems / I. I. Artobolevsky, A. N. Bogolyubov, V. V. Bolotin [et al.] (edited by V. V. Bolotin). - Moscow: Mashinostroenie, - 1978, - 352 p.

[2]Kapustin N.Yu., Moiseev E.I. The basis property in L_p of the systems of eigenfunctions corresponding to two problems with a spectral parameter in the boundary condition // Differential Equations, – 2000. v. 36, no. 10, – 1498–1501.

condition. N.B Kerimov and R.G. Poladov[3], Z.S. Aliev and A.A. Dunyamalieva[4] investigated the Sturm-Liouville problem with a spectral parameter in both boundary conditions and established sufficient conditions for the basis property of systems of root functions in space $L_p, 1 < p < \infty,$ after removing two functions.

N.B. Kerimov and Z.S. Aliev[5], Z.S. Aliev[6,7] studied the spectral properties of boundary value problems for ordinary differential equations of the fourth order (completely regular Sturmian systems) with a spectral parameter in one of the boundary conditions. They established necessary and sufficient conditions under which the system of root functions of these problems forms a basis in $L_p, 1 < p < \infty,$ after removing one function.

Thus, it is relevant to study the spectral properties of regular Sturmian systems with a spectral parameter in one of the boundary conditions and completely regular Sturmian systems with a spectral parameter in two of the boundary conditions.

Object and subject of the study. The object of the study is ordinary differential operators with a spectral parameter in the boundary conditions, and the subject of thestudy is the oscillatory properties of eigenfunctions and basic properties of root functions.

Goal and tasks of the study. The main goal and tasks of the dissertation is to study the location of eigenvalues on the real axis, the structure of root subspaces, the oscillation properties of eigenfunctions and

[3]Kerimov, N.B., Poladov, R.G. Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions // Doklady Mathematics, 2012. v. 85, no. 1, – 18–13.

[4]Aliev, Z.S., Dunyamalieva, A. A. Defect basis property of a system of root functions of a Sturm–Liouville problem with spectral parameter in the boundary conditions \\ Differential Equations, – 2015. v. 51, no. 10, – 1249–1266.

[5]Kerimov N.B., Aliev Z.S. On the basis property of the system of eigenfunctions of a spectral problem with a spectral parameter in the boundary condition // Differential Equations, – 2007, v. 43, no. 7, – 905–915.

[6]Aliyev, Z.S. Basis properties of a fourth order differential operator with a spectral parameter in the boundary condition // Central European Journal Mathematics, – 2010, v. 8, no. 2, – p. 378–388.

[7]Aliev Z.S. Basis properties in L p of systems of root functions of a spectral problem with spectral parameter in a boundary condition // Differential Equations, -- 2011, v. 47, no. 6, – 766–777.

the basis properties in space $L_p, 1<p<\infty$, of the system of root functions of regular and completely regular Sturmian systems, the boundary conditions of which contain a spectral parameter.

İnvestigation methods. The dissertation work uses methods of mathematical analysis, differential equations, functional analysis, complex analysis, theory of operators in a space with an indefinite metric, spectral theories of linear differential operators.

Basic statements to be defended. The following basic provisions are brought up for defense:

– to study the structure of the root subspaces and the oscillatory properties of eigenfunctions of a fourth order regular Sturmian system in the presence of a potential;

– find a general characteristic of the location of eigenvalues on the real axis, determine the multiplicities of all eigenvalues, completely investigate the oscillatory properties of eigenfunctions, and study the basis properties of root functions in the space $L_p, 1<p<\infty$, of a fourth order regular Sturmian system in the presence of a potential and with a spectral parameter in one of the boundary conditions;

– find a general characteristic of the location of eigenvalues on the real axis, study the oscillatory properties of eigenfunctions and their derivatives, obtain asymptotic formulas for the eigenvalues and eigenfunctions, and establish sufficient conditions for the basis property of systems of root functions with two remote functions in the space $L_p, 1<p<\infty$, of a completely regular Sturmian system of fourth order with spectral parameter contained in two boundary conditions.

Scientific novelty of the study. In the dissertation, the following main results were obtained:

for completely regular Sturmian systems in the presence of a potential

– studied the structure of root subspaces and oscillatory properties of all eigenfunctions;

for regular Sturmian systems in the presence of a potential and with a spectral parameter in one of the boundary conditions

– found a general characteristic of the location of eigenvalues on the real axis;

– the oscillatory properties of the eigenfunctions have been completely studied;

– the basis property of the system of eigenfunctions in the space $L_p, 1 < p < \infty,$ after removing any arbitrary function is proved;

for completely regular Sturmian systems with a spectral parameter in two of the boundary conditions

– the location of eigenvalues on the real axis has been studied;

– the oscillatory properties of the eigenfunctions and their derivatives are completely investigated;

– obtained asymptotic formulas for eigenvalues and eigenfunctions;

– sufficient conditions are established for the basis property of systems of root functions in the space $L_p, 1 < p < \infty,$ after removing two functions.

Theoretical and practical value of the study. The results obtained in the dissertation are mainly theoretical in nature. They can be used in the study of various issues in the spectral theory of differential operators, in the study of various processes in mechanics and physics.

Approbation and application. The results obtained in the dissertation by the author were reported at seminars held at the Ganja State University at the Department of Mathematical Analysis (headed by Ass. Prof. A.M. Huseynov), at the Baku State University at the Department of Mathematical Analysis (headed by Prof. S.S. Mirzoev) , at the Khazar University at the Department of Mathematics (headed by Prof. N.B. Kerimov), at the IMM of the National Academy of Sciences of Azerbaijan at the Department of Differential Equations (headed by Prof. A.B. Aliyev), at the International Conference dedicated to the 55-th anniversary of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan "Actual problems of Mathematics and Mechanics" (Baku, 2014), at the International Conference "Areas of Application of Mathematics and ICT. New Learning Technologies" (Ganja, 2014), at the International Conference on the Development of Mathematical Sciences (AMS 2015) (Antalya, Turkey, 2015), at the I International Scientific Conference of Young Scientists (Ganja, 2016), at the International Voronezh Winter Mathematical School "Modern Methods of Function Theory and Related Problems" (Voronezh, Russia, 2021).

The author's personal contribution lies in the formulation of the research goal. All research results obtained belong to the author.

Author's publications. Publications recommended by the Higher Attestation Commission under the President of the Republic of Azerbaijan -

7 (including 2 WOS, 2 SCOPUS) and conference materials - 5 (all conferences are international, 2 of them were conducted abroad).

The institution where the dissertation work was performed. The work was performed at the chair of "Mathematical analysis" of Ganja State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately). Total volume of the dissertation work– 205750 signs (title page – 381 signs, contents – 2254 signs, introduction – 54000 signs, chapter I – 54000 signs, chapter II – 94000 signs, conclusion – 1115 signs). The list of references consists of 77 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation work consists of an introduction, 2 chapters (with 11 paragraphs), a conclusion and a list of used literature.

The first chapter, consisting of fourth sections, is devoted to the study of oscillatory and basis properties of the system of eigenfunctions of regular Sturmian systems for ordinary differential equations of fourth order with a spectral parameter in the boundary condition. To study these properties, we study the oscillatory properties of some regular Sturmian systems (without the inclusion of a spectral parameter in the boundary conditions). The existence and uniqueness of the solution to the initial-boundary value problem for these regular Sturmian systems is proved.

In 1.1, we study the oscillatory properties of a regular Sturmian system

$$\ell_r(y) \equiv (p(x)y'')'' - (q(x)y')' + r(x)y(x) = \lambda\tau(x)y, \, 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{2b}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{2c}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{2d}$$

where $\lambda \in C$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, $p(x)$ is positive function and has an absolutely continuous derivative on $[0, l]$, $q(x)$ is non-negative and absolutely continuous function on $[0, l]$, $r(x)$ is continuous function on $[0, l]$, $\tau(x)$ is positive and continuous function on $[0, l]$, $\alpha, \beta, \gamma, \delta$ are real constants such that $\alpha, \beta, \gamma \in [0, \pi/2]$, $\delta \in [0, \pi)$.

Problem (1), (2) arises when the variables are separated in a dynamic boundary value problem describing small transverse vibrations of an inhomogeneous rod subjected to the action of an axial (or longitudinal) force.

The eigenvalues of completely regular Sturmian systems are real and form an unboundedly nondecreasing sequence; moreover, in the case $r(x)\equiv 0,$ they are positive and simple, and the corresponding eigenfunctions have usual oscillatory properties. In the case when $r(x)$ does not identically vanish, in any interval that makes up a part of the interval $[0, 1],$ the eigenvalues are simple, with the exception, perhaps, of the first $m,$ and the corresponding eigenfunctions have usual oscillatory properties (here m is some natural number greater than two).

Note that problem (1), (2) for $\delta\in[0,\pi/2],$ except for the case $\beta=\delta=\pi/2,$ is a completely regular Sturmian system, and in the case $\delta\in[\pi/2,\pi)$ it is a regular Sturmian system. The oscillatory properties of eigenfunctions of problem (1), (2) and their derivatives were studied in detail by D.O. Banks and G.J. Kurowski[8] for $r\equiv 0$ and $\delta\in[0,\pi/2]$, and by Z.S. Aliev[9] for $\delta\in[\pi/2,\pi)$. Only Z.S. Aliev[10] completely investigated the oscillatory properties of eigenfunctions corresponding to the first m eigenvalues of completely regular Sturmian systems.

Let $\delta_0=\pi/2$ if $\beta\in[0,\pi/2),$ $\delta_0=arctg\ F_0(0)$ if $\beta=\pi/2,$ $\mathrm{N}_0=\mathrm{N},$ if $\delta\in[0,\delta_0)$ and $\mathrm{N}_0=\mathrm{N}\setminus\{1\}$ if $\delta\in[\delta_0,\pi),$ where $F_0(\lambda)=\dfrac{T\vartheta(1,\lambda)}{\vartheta(1,\lambda)}$ and the function $\vartheta(x,\lambda)$ is a solution of problem (1), (2a)-(2c) with $r\equiv 0.$

This section is devoted to the study of the structure of root

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[8] Banks, D.O., Kurowski G.J. A Prufer transformation for the equation of the vibrating beam // Transactions of the American Mathematical Society, – 1974. v.199, no.1, – p. 203–222.

[9] Aliyev, Z.S. Structure of root subspaces and oscillation properties of eigenfunctions of one fourth order boundary value problem // – Baku: Azerbaijan Journal of Mathematics, – 2014. v. 4, no. 2, – p. 108-121.

[10] Aliyev Z.S. Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order // – Moscow: Sbornik Mathematics, – 2016. v. 207, no. 12, – p. 1625–1649.

subspaces and oscillation properties of eigenfunctions corresponding to the first m eigenvalues of problem (1), (2) for $\delta \in [\pi/2, \pi)$.

The following theorem is a main result of Section 1.1.

Theorem 1. *For fixed* α, β, γ *the spectrum of problem (1), (2) for* $\delta \in [0, \pi)$ *consists of real and simple eigenvalues that form an unbounded increasing sequence* $\{\lambda_n(\delta)\}_{n=1}^{\infty}$. *Moreover, the eigenfunction* $y_{n.\delta}(x)$ *corresponding to the eigenvalue* $\lambda_n(\delta)$ *for* $n \in \mathrm{N}_0$ *has exactly* $n-1$ *simple zeros in the interval* $(0, l)$.

In 1.2 we study the existence, uniqueness, and main properties of a solution to problem (1), (2a) - (2c).

Theorem 2. *For each fixed* $\lambda \in C$ *there is a unique nontrivial solution* $y(x,\lambda)$ *of problem (1), (2a)-(2c) up to a constant factor.*

For each fixed $x \in [0, l]$ the function $y(x,\lambda)$) is an entire function of parameter λ. Note that the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi/2)$, $n \in \mathrm{N}$, of problem (1), (2) are zeros of entire functions $y(1,\lambda)$ and $Ty(1,\lambda)$), respectively. It is obvious that the function $F_r(\lambda) = Ty(l,\lambda)/y(l,\lambda)$ is defined on the set $B = (C \setminus R) \cup \bigcup\limits_{n=1}^{\infty} B_n$, where $B_n = (\lambda_{n-1}(0), \lambda_n(0))$, $n \in \mathrm{N}$, and $\lambda_0(0) = -\infty$, and is a meromorphic function of finite order, and $\lambda_n(0)$, and $\lambda_n(\pi/2)$, $n \in \mathrm{N}$, are poles and zeros of this function, respectively.

Lemma 1. *The relation holds:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{y^2(l,\lambda)} \int_0^l \tau\, y^2(x,\lambda)dx, \ \lambda \in B. \tag{4}$$

The following asymptotic formula holds:

$$F_r(\lambda) = -\left(\sqrt{2}\right)^{1-2sgn\gamma} (p(1)\tau^3(1))^{\frac{1}{4}} \sqrt[4]{|\lambda|^3} \left(1 + O\left(1/\sqrt[4]{|\lambda|}\right)\right), \ \lambda \to -\infty, \tag{5}$$

whence implies that

$$\lim_{\lambda \to -\infty} F_r(\lambda) = -\infty. \tag{6}$$

Now we investigate the question on the number of zeros of the function $y(x,\lambda)$ contained in the interval $(0, l)$.

Lemma 2. *Each zero* $x(\lambda)$ *of the function* $y(x,\lambda)$ *is a simple and continuously differentiable function of* $\lambda \in [-\infty, +\infty)$.

Let μ be a real eigenvalue of Eq. (1) under the boundary conditions $y(0)=y'(0)=y''(0)=0$, (2c) if $\beta=0$, or $y(0)=Ty(0)=0$, (2a), (2c) if $\beta\in(0,\pi/2]$. The oscillation index of the eigenvalue μ is the difference between the numbers of zeros contained in the interval $(0,\, l)$ of the function $y(x,\lambda)$ for $\lambda\in(\mu-\varepsilon,\mu)$ and $\lambda\in(\mu,\mu+\varepsilon)$, where $\varepsilon>0$ is a sufficiently small number.

Lemma 3. *There is a number $\xi<0$ such that the real eigenvalues ξ_k, $k=1, 2, \dots,$ of the spectral problem (1), $y(0)=y'(0)=y''(0)=0$, (2c) for $\beta=0$ or $y(0)=Ty(0)=0$, (2a), (2c) for $\beta\in(0,\pi/2]$ are contained in the interval $(-\infty,\, \xi)$, are simple, have an oscillation index of 1, and form an infinitely decreasing sequence.*

Let $\kappa(\lambda)$ be the number of zeros of the function $y(x,\lambda)$ which are contained in the interval $(0,l)$.

Lemma4. *If $\lambda>\lambda_1(\delta_0)$ and $\lambda\in(\lambda_{n-1}(0),\lambda_n(0)]$, $n\in\mathrm{N}$, then $\kappa(\lambda)=n-1$, and if $\lambda\le\lambda_1(\delta_0)$, then $\kappa(\lambda)=\sum\limits_{\xi_k\in(\lambda,\lambda_1(\delta_0))} i(\xi_k)$, where $i(\xi_k)$ is the oscillation index of the eigenvalue ξ_k.*

In 1.3, we consider the Sturmian system (1), (2a) - (2.c) and

$$(a\lambda+b)y(l)-(c\lambda+d)Ty(l)=0, \tag{7}$$

where a, b, c, d are real constants such that $\sigma=bc-ad>0$. The structure of the root subspaces and the oscillatory properties of eigenfunctions of this problem are studied.

For $c\ne0$ let $N_0\in\mathrm{N}$ be an integer such that

$$\lambda_{N_0-1}(0)<-d/c\le\lambda_{N_0}(0).$$

Theorem 3. *The eigenvalues of the spectral problem (1), (2.a)-(2.c), (7) are real, simple, and form an infinitely increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$. The eigenfunction $y_n(x)$, $n\in\mathrm{N}$, corresponding to the eigenvalue λ_n has the following oscillatory properties: a) if $c=0$, then $y_n(x)$ for $n\ge2$ has exactly $n-1$ simple zeros in the interval $(0,l)$, the function $y_1(x)$ for $\lambda_1>\lambda_1(\delta_0)$ has no zeros in the interval $(0,l)$, for $\lambda_1\le\lambda_1(\delta_0)$ has $m(\lambda_1)$ simple zeros in the interval $(0,\, l)$; b) if $c\ne0$, then in case $\lambda_n\ge\lambda_1(\delta_0)$ the function $y_n(x)$ for $n\le N_0$ has exactly $n-1$ simple zeros, for $n>N_0$ has exactly $n-2$ simple zeros in the interval $(0,\, l)$, in case*

$\lambda_n < \lambda_1(\delta_0)$ *for* $n=1$ *or* $n=2$ *the function* $y_n(x)$ *has* $m(\lambda_n)$ *simple zeros in the interval* $(0, l)$.

In 1.4, the basis properties in the space $L_2((0,l),\tau)$ of eigen-functions of problem (1), (2.a)-(2.c), (7) are investigated.

Let $H = L_2((0,l),\tau)\oplus C$ be a Hilbert space with scalar product

$$(\hat{y},\hat{\vartheta}) = (\{y,k\},\{\vartheta,t\}) = (y,u)_{L_{2,\tau}} + |\sigma|^{-1}k\bar{t},$$

where $(y,\vartheta)_{L_{2,\tau}} = \int\limits_o^l \tau(x)y(x)\overline{\vartheta(x)}\,dx.$ We define an operator

$$L\hat{y} = L\{y,k\} = \{\tau^{-1}(x)\ell_r(y)(x), dTy(l) - by(l)\}$$

on the domain

$$D(L) = \{\hat{y} = \{y(x),k\}\in H : y(x)\in W_2^4(0,l),\ \ell_r(y)(x)\in L_2(0,l),$$
$$y\in B.C.,\ k = ay(l) - cTy(l)\},$$

which is everywhere dense in H, where $B.C.$ is the set of functions satisfying the boundary conditions (2a)-(2c). It is known that the problem (1), (2. a)-(2. c), (7) reduced to the following eigenvalue problem

$$L\hat{y} = \lambda\hat{y},\ \hat{y}\in D(L). \tag{8}$$

In this case, the eigenvalues $\lambda_n,\ n\in \mathrm{N},$ of problems (1), (2.a)-(2. c), (7) and (8) coincide, and between the eigenvectors, there is a one-to-one correspondence

$$y_n(x)\leftrightarrow \hat{y}_n = \{y_n(x),\ k_n\},\ k_n = ay_n(l) - cTy_n(l).$$

Theorem 4. *L is a self-adjoint, discrete, lower-semibounded operator in H. The system of root vectors $\{\hat{y}_n\}_{n=1}^{\infty}$, $\hat{y}_n = \{y_n(x),\ k_n\}$ of this operator forms an orthogonal basis in H.*

Denote: $\delta_n = (\hat{y}_n,\hat{y}_n) = \| y_n \|_{L_{2,\tau}}^2 + \sigma^{-1}k_n^2,\ n\in \mathrm{N}.$ By the condition $\sigma > 0$ we have

$$\delta_n > 0 \quad \text{и} \quad k_n = ay_n(l) - cTy_n(l) \neq 0,\ n\in \mathrm{N}. \tag{9}$$

In view of (9), by Theorem 4 the system $\{\hat{\vartheta}_n\}_{n=1}^{\infty}$, $\hat{\vartheta}_n = \delta_n^{-1/2}\hat{y}_n$, of eigenvectors of problem (8) forms an orthonormal basis in H.

Theorem 5. *Let s be an arbitrary fixed natural number. Then the system $\{y_n(x)\}_{n=1,n\neq l}^{\infty}$ forms a basis in space $L_p((0,\ l),\tau),\ 1<p<\infty,$ which is an unconditional basis in $L_2((0,\ l),\tau)$. In this case the system $\{u_n(x)\}_{n=1,n\neq l}^{\infty}$ adjoint to the system $\{y_n(x)\}_{n=1,n\neq l}^{\infty}$ is determined by the relation:*

$$u_n(x) = y_n(x) - \frac{\delta_l k_n}{\delta_n k_l} y_l(x).$$

In Chapter II, spectral problems are considered that describe bending vibrations of a homogeneous Euler-Bernoulli beam, in the cross sections of which a longitudinal force acts, the left end of which is fixed, and at the right end an inertial load is concentrated and a tracking force acts, and this end is either fixed elastically or freely. Spectral properties, including basis properties in the space $L_p(0, 1)$, $1 < p < \infty$, of subsystems of eigenfunctions of these problems, are studied.

Section 2.1 gives the problem statement. Consider an Euler-Bernoulli beam of length L with a rectilinear axis of variable non-twisted section, performing bending vibrations in the Oxz plane. Free bending vibrations of a homogeneous beam, in the sections of which the longitudinal force $\overline{Q}(X)$ acts, is described by the equation

$$EJ\frac{\partial^4 W(X,t)}{\partial X^4} - \frac{\partial}{\partial X}\left(\overline{Q}(X)\frac{\partial W(X,t)}{\partial X}\right) + \rho F\frac{\partial^2 W(X,t)}{\partial t^2} = 0,$$

where J is the moment of inertia of the cross-section relative to Oy, EJ is a bending stiffness of the beam, $W(X,t)$ is the deflection of the current point of the rod axis, ρ is the density of the beam, and F is the cross-sectional area of the beam.

If the left end of the beam is fixed, and at the right en a load of mass m and inertia I is concentrated, then the boundary conditions are written in the following form

$$W(0,t) = 0, \quad \frac{\partial W(0,t)}{\partial x} = 0, \quad EJ\frac{\partial^2 W(L,t)}{\partial X^2} = -I\frac{\partial^3 W(L,t)}{\partial X \partial t^2},$$

$$EJ\frac{\partial^3 W(L,t)}{\partial X^3} - \overline{Q}(L)\frac{\partial W(L,t)}{\partial X} = m\frac{\partial^2 W(L,t)}{\partial t^2} = 0.$$

We introduce the notation $x = X/L$ and $w = W/L$, and write the equations of free bending vibrations of a homogeneous beam with constant stiffness and the boundary conditions in the form

$$EJ\frac{\partial^4 w(x,t)}{\partial x^4} - \frac{\partial}{\partial x}\left(Q(x)\frac{\partial w(x,t)}{\partial x}\right) + \frac{\rho F L^4}{EJ}\frac{\partial^2 w(x,t)}{\partial t^2} = 0,$$

$$w(0,t) = 0, \quad \frac{\partial w(0,t)}{\partial x} = 0, \quad EJ\frac{\partial^2 w(1,t)}{\partial x^2} = -\frac{IL}{EJ}\frac{\partial^3 w(1,t)}{\partial x \partial t^2},$$

$$\frac{\partial^3 w(1,t)}{\partial x^3} - Q(1)\frac{\partial w(1,t)}{\partial x} = \frac{mL^3}{EJ}\ \frac{\partial^2 W(1,t)}{\partial t^2} = 0,$$

where $Q(x) = \dfrac{L^2}{EJ}\overline{Q}(Lx)$.

We denote $\rho FL^4\omega^2/EJ$ by λ. Then the problem of free bending vibrations of a beam by replacing $w(x,t) = y(x)\cos\omega t$ is reduced to the following eigenvalue problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{10}$$

$$y(0) = y'(0) = 0, \tag{11}$$

$$y''(1) - a_1\lambda y'(1) = 0, \tag{12}$$

$$Ty(1) - a_2\lambda y(1) = 0, \tag{13}$$

where $q(x) \equiv Q(x)$, $Ty \equiv y''' - qy'$, a $a_1 = I/\rho FL^3$, $a_2 = -m/FL$. Note that the conditions $q(x) > 0, x \in [0, 1], a_1 > 0,\ a_2 < 0,$ are satisfied. In addition, we assume that the function $q(x)$ is absolutely continuous on [0, 1].

In 2.2, some auxiliary facts and statements are given.

We introduce a boundary condition

$$y'(1)\cos\gamma + y''(1)\sin\gamma = 0, \tag{14}$$

where $\gamma \in [0, \pi/2]$.

In the work of N. B. Kerimov and Z. S. Aliev[5], it iwas proved that the eigenvalues of the boundary value problem (10), (11), (14), (13) are real and simple, and form an infinitely increasing sequence $\{\lambda_n(\gamma)\}_{n=1}^{\infty}$ such that $\lambda_n(\gamma) > 0,\ n \in \mathrm{N}$. Moreover, the eigenfunction $y_{n,\gamma}(x), n \in \mathrm{N},$ corresponding to the eigenvalue $\lambda_n(\gamma)$, has exactly $n-1$ simple zeros in $(0, 1)$.

In 2.3, we prove the existence of the uniqueness of the solution to the initial-boundary value problem (10)-(12) and study its main properties.

Theorem 6. *For each fixed* $\lambda \in C$ *there is a unique nontrivial solution* $y(x,\lambda)$ *to the problem (10)-(12) up to a constant factor.*

Remark 1. Without loss of generality, we can assume that for each fixed $x \in [0, 1]$ the function $y(x,\lambda)$ is an entire function of λ.

It should be noted that the function $y(x,\lambda)$ plays a fundamental role in the study of the spectral properties of the problem (10)-(13).

Lemma 5. *The zeros of the function* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *contained in the interval* $(0, 1]$ *are simple and continuously differentiable functions of* *parameter* λ.

We introduce the function $H(x,\lambda)=y'(x,\lambda)/y''(x,\lambda)$. By Theorem 6, Lemma 5, and Remark 1 for each fixed $x\in[0,1]$ the function $H(x,\lambda)$ is a meromorphic function of finite order of the parameter λ.

Let $K_n=(\lambda_{n-1}(0),\lambda_n(0)),\ n=1,2,\ldots,$ where $\lambda_0(0)=-\infty$.

Note that, the function $G(\lambda)=1/H(1,\lambda)=y''(1,\lambda)/y'(1,\lambda)$ is defined for the values $\lambda\in K\equiv(C\setminus R)\cup\left(\bigcup\limits_{n=1}^{\infty}K_n\right)$, where $\lambda_n(\pi/2)$ and $\lambda_n(0),\ n\in\mathrm{N}$, are zeros and poles of this function, respectively.

Lemma 6. *The following formula holds:*

$$\frac{dG}{d\lambda}=-\frac{1}{y'^2(1,\lambda)}\left\{\int\limits_0^1 y^2(x,\lambda)\,dx-a_2y^2(1,\lambda)\right\},\ \lambda\in K. \tag{15}$$

Lemma 7. *The following relation holds:*

$$\lim_{\lambda\to-\infty}G(\lambda)=+\infty\,. \tag{16}$$

The following comparison type theorem is valid.

Lemma 8. *Let* $0<\xi<\eta$. *If* $y'(x,\xi)$ *has* m *zeros in the interval* $(0,1)$, *then* $y'(x,\eta)$ *has at least* $m\,t$ *zeros in the same interval.*

By $\tau(\lambda)$ and $s(\lambda)$ we denote the number of zeros contained in the interval $(0,1)$ of functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Theorem 7. *The functions* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *have the following oscillatory properties: if* $\lambda\in(0,\lambda_1(0)]$, *then* $\tau(\lambda)=s(\lambda)=0$, *if* $\lambda\in(\lambda_{n-1}(0),\lambda_n(\pi/2))$ *and* $n\geq 2$, *then either* $\tau(\lambda)=n-2$ *or* $\tau(\lambda)=n-1$, *and if* $\lambda\in[\lambda_n(\pi/2),\lambda_n(0)]$ *and* $n\geq 2$, *then* $\tau(\lambda)=n-1$, *if* $\lambda\in(\lambda_{n-1}(0),\lambda_n(0)]$ *and* $n\geq 2$, *then* $s(\lambda)=n-1$.

Remark 2. If $\lambda\in(\lambda_{n-1}(0),\lambda_n(\pi/2))$ and is sufficiently close to $\lambda_{n-1}(0)$, then $\tau(\lambda)=n-2$, and if λ is sufficiently close to $\lambda_n(\pi/2)$, then $\tau(\lambda)=n-1$.

In 2.4, we study the structure of root subspaces and the oscillation properties of eigenfunctions of problem (10)-(13).

Remark 3. If λ is an eigenvalue of problem (10)-(13), then $y'(1,\lambda)\neq 0$.

Theorem 8. *The eigenvalues of the spectral problem (10)-(13) are* *real and simple, and form an unboundedly increasing sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $\lambda_n > 0,\ n \in \mathrm{N}.$ *The corresponding eigenfunctions and their derivatives have the following oscillatory properties: (i) the eigenfunction* $y_n(x)$ *corresponding to the eigenvalue* λ_n *for* $n=1$ *has no zeros, and for* $n \geq 2$ *has either* $n-2$ *or* $n-1$ *simple zeros in* $(0,1)$; *(ii) the function* $y_n'(x),\ n \in \mathrm{N},$ *has exactly* $n-1$ *simple zeros in the interval* $(0,1).$

Theorem 9. *The following asymptotic formulas hold:*

$$\sqrt{\lambda_n} = \left(n - 3/2\right)\pi + O\left(1/n\right), \tag{17}$$

$$\begin{aligned} y_n(x) = \sin\left(n-3/2\right)\pi x - \cos\left(n-3/2\right)\pi x + e^{-(n-3/2)\pi x} + \\ + (-1)^n e^{-(n-3/2)\pi(1-x)} + O\left(1/n\right), \end{aligned} \tag{18}$$

where the relation (20) holds uniformly for $x \in [0, 1].$

In 2.5, we study the basis properties in the space $L_p(0,1),\ 1 < p < \infty,$ of subsystems of eigenfunctions of the boundary value problem (10)-(13). This problem is reduced to a spectral problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus C^2$ with the scalar product

$$(\hat{y}, \hat{\vartheta}) = (\{y, m, k\}, \{\vartheta, s,\ t\}) = (y, \vartheta)_{L_2} + |\, a_1 \,|^{-1}\, m\bar{s} + |\, a_2 \,|^{-1}\, k\,\bar{t}, \tag{19}$$

where $L\hat{y} = L\{y, m, k\} = \{\ (Ty)'(x), y''(1), Ty(1)\},$ defined on the set

$$D(L) = \{\hat{y} = \{y(x), m\} \in H\ :\ y(x) \in W_2^4(0,\ 1),\ (Ty)'(x) \in L_2(0,\ 1),$$
$$y(0) = y'(0) =\ 0,\ m = a_1 y'(1),\ k = a_2 y(1)\},$$

which is everywhere dense in H . Note that the operator L is well defined in H. In this case, the spectral problem (10)-(13) takes the form

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L).$$

Theorem 10. *The operator* L *is self-adjoint discrete semi-bounded in* H. *The system* $\{\hat{y}_n\}_{n=1}^{\infty},\ \hat{y}_n = \{y_n(x),\ m_n,\ k_n\},\ m_n = a_1 y_n'(1),\ k_n = a_2 y_n(1),$ *of eigenvectors of this operator forms an orthogonal basis in* H.

Denote: $\delta_n = (\hat{y}_n, \hat{y}_n).$ In view of conditions $a_1 > 0, a_2 < 0,$ by (19) we have

$$\delta_n = \|\, y_n \,\|_{L_2}^2 + a_1 y'^2(1) - a_2 y^2(1) > 0. \tag{20}$$

Hence, by theorem 10, the system of eigenvectors $\{\hat{\vartheta}_n\}_{n=1}^{\infty},\ \ \hat{\vartheta}_n = \delta_n^{-1/2}\,\hat{y}_n$, of the operator L forms an orthonormal basis (Riesz basis) in H.

Let be $r, l\ (r \neq l)$ be an arbitrary fixed natural numbers and

$$\Delta_{r,l}=\begin{vmatrix} y_r'(1) & y_l'(1) \\ y_r(1) & y_l(1) \end{vmatrix}. \tag{21}$$

Theorem 11. *Let* $\Delta_{r,l}\neq 0.$ *Then the system* $\{y_n(x)\}_{n=1,n\neq r,l}^{\infty}$ *of eigenfunctions of problem (10)-(13) forms a basis in the space* $L_p(0,1),$ $1<p<\infty,$ *and this basis is an unconditional basis for* $p=2.$ *Now let* $\Delta_{r,l}=0.$ *Then the system* $\{y_n(x)\}_{n=1,n\neq r,l}^{\infty}$ *incomplete and nonminimal in* $L_p(0,1),$ $1<p<\infty.$

Remark 4. By Theorem 8 and Remark 2, for each $n\in\mathrm{N}$ and for any small $\delta>0,$ the following relations hold:

$$\tau(\lambda)=n-1 \text{ if } \lambda\in(\lambda_n(0),\lambda_n(0)+\delta),$$
$$\tau(\lambda)=n \text{ if } \lambda\in(\lambda_{n+1}(\pi/2)-\delta,\lambda_{n+1}(0)).$$

For every natural $n\in\mathrm{N}$ there is a unique $\eta_n\in(\lambda_{n+1}(0),\ \lambda_{n+1}(\pi/2))$ such that $y(1,\eta_n)=0.$ Then, by the last two relations, we get $\tau(\lambda)=n-1$ for $\lambda\in(\lambda_n(0),\eta_n],$ $\tau(\lambda)=n$ for $\lambda\in(\eta_n,\lambda_{n+1}(0)).$ Moreover, the following asymptotic formula holds:

$$\sqrt[4]{\eta_n}=n\pi+O(1/n). \tag{22}$$

Remark 5. By virtue of the asymptotics (17) and (22), for each a_1 there exists a natural number $n^{(1)}=n_{a_1}\geq 2$ such that for $n\geq n^{(1)}$ we have $\lambda_{n-1}(0)<\lambda_n<\eta_{n-1}<\lambda_n(\pi/2).$ Then $\tau(\lambda)=n-2$ for $n\geq n^{(1)}.$ Hence it follows from Lemmas 6, 7 and formula (17) that with increasing a_1 the number $n^{(1)}$ decreases. Consequently, there exists a number $a_1^{(1)}>0$ such that $n^{(1)}=2$ for $a_1>a_1^{(1)}.$

Remark 6. Let $r\geq 2$ and $a_{1,r}$ be a positive number such that $a_{1,r}\eta_{r-1}=G(\eta_{r-1}).$ Then $\lambda_r=\eta_{r-1}$ for $a_1=a_{1,r}.$ Therefore, by Lemma 6, we have $\lambda_{r-1}(0)<\eta_{r-1}<\lambda_r<\lambda_r(\pi/2)$ for $a_1<a_{1,r}.$ Hence $r<n^{(1)}$ for $a_1\leq a_{1,r}.$

Theorem 12. *If* $r=1,\, l\geq n^{(1)}$ *or* $r\geq 2,\, a_1\leq a_{1,r},\, l\geq n^{(1)},$ *then the system* $\{y_n(x)\}_{n=1,n\neq r,l}^{\infty}$ *forms a basis in space* $L_p(0,1),$ $1<p<\infty,$ *which is an unconditional basis in* $L_2(0,1)$.

In 2.6, we study the properties of eigenvalues and eigenfunctions of the spectral problem

$$y^{(4)}(x)-(q(x)y'(x))'=\lambda y(x),\ 0<x<1,$$
$$y(0)=0,\ y'(0)=0,\ y''(1)-(a_1\lambda+b_1)y'(1)=0,\ Ty(1)-a_2\lambda y(1)=0, \tag{23}$$

where $b_1\neq 0$. In case $b_1<0$, this problem describes the bending vibrations of a homogeneous rod with a constant stiffness, in the cross-sections of which a longitudinal force acts, the left end of which is fixed, and the right end is elastically fixed, and at this end an inertial load is concentrated and a tracking force acts.

Lemma 9. *As $\lambda\le 0$ varies, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ can only lose zeros or gain only by these zeros leaving or entering the interval $(0,1)$ through its endpoint $x=0$.*

Let μ be the real eigenvalue of the following spectral problem

$$\ell(y)(x)=\lambda y(x),\ 0<x<1,$$
$$y(0)=y'(0)=y''(0)=Ty(1)-a_2\lambda y(1)=0. \tag{24}$$

Lemma 10. *There is $\zeta<0$ such that all real eigenvalues μ_k, $k=1,2,\ .\ .\ .\ ,$ of problem (26) are simple, contained in the ray $(-\infty,\zeta)$, form an unboundedly decreasing sequence, have an oscillation index of 1, and admit the asymptotics $\mu_k=-4\left(k\pi+\pi/2\right)^4+o(k^4)$.*

Let $\lambda<0$ and $i(\mu_k)$ be the oscillation index of the eigenvalue $\mu_k, k\in \mathrm{N}$, of problem (24). Then we have

$$s(\lambda)=\tau(\lambda)=\sum_{\mu_k\in(\lambda,0)} i(\mu_k). \tag{25}$$

Let us determine $N_1\in\mathrm{N}$ from the inequality

$$\lambda_{N_1-1}(\pi/2)<-b_1/a_1\le\lambda_{N_1}(\pi/2).$$

Theorem 13. *The eigenvalues of problem (23) are real and simple, and form an infinitely increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n>0$ for $n\ge 2$. The corresponding eigenfunctions $y_1(x), y_2(x),\ .\ .\ .\ ,\ y_n(x),\ .\ .\ .\ ,$ and their derivatives have the following oscillatory properties: i) if $N_1=1$, then the functions $y_1(x)$ u $y_1'(x)$ have no zeros in the case $\lambda_1\ge 0$, have $\sum_{\mu_n\in(\lambda_1,0)} i(\mu_n)$ simple zeros in the interval $(0,1)$ in the case $\lambda_1<0$, the function $y_n(x)$ for $n\ge 2$ has either $n-2$ or $n-1$ simple zeros, and the function $y_n'(x)$ has exactly $n-1$ simple zeros in the interval $(0,1)$; ii) if*

$N_1 > 1$, *then the function* $y_n(x)$ *for* $n < N_1$ *has exactly* $n-1$ *simple zeros, for* $n \ge N_1$ *either* $n-2$ *or* $n-1$ *simple zeros, and the function* $y_n'(x)$ *has* $n-1$ *simple zeros in the interval* $(0, 1)$.

Note that problem (23) reduces to an eigenvalue problem for a self-adjoint operator $\tilde{L}$ in a Hilbert space $H = L_2(0,1) \oplus C^2$ with scalar product (19), where the operator $\tilde{L}$ is defined as follows:

$$\hat{L}\hat{y} = L\{y, m, k\} = \{\, (Ty)'(x), y''(1) - b_1 y'(1), Ty(1)\},$$

$$D(L) = \{\hat{y} = \{y(x), m, k\} \in H \,:\, y(x) \in W_2^4(0, 1),\ (Ty)'(x) \in L_2(0, 1),$$

$$y(0) = y'(0) = \ 0,\ m = a_1 y'(1),\ k = a_2 y(1)\}.$$

Remark 7. It follows from (17) that there exists a natural number $\tilde{n} \ge 2$ such that the function $y_n(x)$ for $n \ge \tilde{n}$ has exactly $n-2$ simple zeros in the interval $(0, 1)$. It is obvious that $\tilde{n} > N_1$.

Theorem 14. *Let* r, l $(r \ne l)$ *be arbitrary fixed natural numbers. If* $b_1 < 0$, $N_1 > 1$, $r < N_1$, $l \ge \tilde{n}$, *or* $b_1 > 0$, $r = 1$, $l \ge \hat{n}$, *then the system* $\{y_n(x)\}_{n=1, n\ne r, l}^{\infty}$ *of eigenfunctions of problem (23) forms a basis in space* $L_p(0, 1)$, $1 < p < \infty$ *(an unconditional basis for* $p = 2$*).*

In 2.7, we study the spectral properties of problem (10)-(13) with $a_1 < 0$ and $a_2 < 0$. In this case L is a closed (non-self-adjoint) operator in H with a compact resolvent.

We define the operator $J : H \to H$ as follows:

$$J\{y, m, k\} = \{y, -m, k\}\,.$$

Lemma 11. J *is a unitary and symmetric operator on* H. *The spectrum of this operator consist of two eigenvalues:* -1 *with multiplicity* 1 *and* $+1$ *with infinite multiplicity.*

By Lemma 12 the operator J generates a Pontryagin space $\Pi_1 = L_2(0, 1) \oplus C^2$ with an inner product (J - metric)

$$[\hat{y}, \hat{\vartheta}] = (\hat{y},\ \hat{\vartheta})_{\Pi_1} = (\{y, m, k\}, \{\vartheta, s, t\})_{\Pi_1} = (y, \vartheta)_{L_2^r} + a^{-1} m\bar{s} - a_2^{-1} k\bar{t}\,. \quad (26)$$

Theorem 15. *The operator* L *is* J *-self-adjoint in* Π_1. *The system of root vectors of this operator forms a Riesz basis (after normalization) in* H.

Theorem 16. *Let* $a_1 < 0$ *and* $a_2 < 0$. *Then all eigenvalues of the* *boundary value problem (10)-(13) are real, simple, and form an unboundedly increasing sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $\lambda_1 < 0$ *and* $\lambda_n > 0,\ n \geq 2.$ *The corresponding eigenfunctions* $y_n(x),\ n \in \mathrm{N},$ *and their derivatives have the following oscillatory properties: functions* $y_n(x)$ *and* $y_n'(x)$ *for* $n \geq 3$ *have exactly* $n-2$ *simple zeros, for* $n = 2$ *have no zeros, and for* $n = 1$ *have exactly* $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ *simple zeros in the interval* $(0, 1).$

Lemma 12. *Let* $\{\hat{\vartheta}_n\}_{n=1}^{\infty},\ \hat{\vartheta}_n = \{\vartheta_n, s_n, t_n\},$ *be the system conjugate to the system* $\{\hat{y}_n\}_{n=1}^{\infty}.$ *Then* $\hat{\vartheta}_n = \delta_n^{-1} \hat{y}_n,\ n \in \mathrm{N}.$

Theorem 17. *Let* $r,\ l\ (r \neq l)$ *be arbitrary fixed natural numbers. If* $\Delta_{r,l} \neq 0,$ *then the system* $\{y_n(x)\}_{n=1}^{\infty}$ *of eigenfunctions of problem (10)-(13) with* $a_1 < 0, a_2 < 0,$ *forms a basis in the space* $L_p(0,1),\ 1 < p < \infty,$ *which is an unconditional basis for* $p = 2,$ *and if* $\Delta_{r,l} = 0,$ *then this system is incomplete and nonminimal in* $L_p(0,1),\ 1 < p < \infty.$

Conclusion

In the dissertation work, regular and completely regular Sturmian systems with a spectral parameter under boundary conditions are considered. The subject of research is relevant, since the problems under consideration are encountered in the study of various processes in mechanics and physics.

In the dissertation, the following main results obtained: for completely regular Sturmian systems in the presence of a potential

– studied the structure of root subspaces and oscillatory properties of all eigenfunctions;

for regular Sturmian systems in the presence of a potential and with a spectral parameter in one of the boundary conditions

– found a general characteristic of the location of eigenvalues on the real axis;

– the oscillatory properties of the eigenfunctions completely studied;

– the basis properties of the system of eigenfunctions in the space $L_p,\ 1 < p < \infty,$ after removing any arbitrary function is proved;

– for completely regular Sturmian systems with a spectral parameter in two of the boundary conditions

– the location of eigenvalues on the real axis studied;

– the oscillatory properties of eigenfunctions and their derivatives are completely investigated;

– obtained asymptotic formulas for eigenvalues and eigenfunctions;

– sufficient conditions are established for the basis property of systems of root functions in the space $L_p, 1 < p < \infty$, after removing two functions.

The main results of the dissertation are published in the following works:

1. Aliev, Z. S., Guliyeva, S. B. Oscillatory properties of eigenfunctions of the equation of transverse vibrations of a rod exposed to axial forces / / Proceedings of the International Conference "Actual Problems of Mathematics and Mechanics" dedicated to the 55th anniversary of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, - Baku: - 2014, - pp. 38-40.
2. Aliev, Z. S., Gulieva, S. B. Some spectral properties of a vibrational boundary value problem with a spectral parameter in a boundary condition / / Proceedings of the International Conference " Spheres of Application of Mathematics and ICT. New learning technologies", part I – - Ganja: - 2014, - pp. 112-115.
3. Aliyev, Z.S., Guliyeva, S.B. Oscillation properties of eigenfunctions of a vibrational boundary value problem // – Baku: Transactions of NAS of Azerbaijan, ser. phys.–tech. math. sci., mathematics, – 2014. v. 34, no. 4, – p. 29–36.
4. Aliyev, Z.S., Guliyeva, S.B. Spectral properties for the equation of vibrating beam // – Baku: Proceedincs of IMM of NAS of Azerbaijan, – 2015, v. 41, no. 1, – p. 135–145.
5. Aliyev, Z.S., Guliyeva, S.B., Some spectral properties of an eigenvalue problem with spectral parameter in the boundary conditions // The Abstract of International Conference on Advancement in Mathematical Sciences, – Antalya: – 05-07 November, – 2015, – p. 112.

6. Guliyeva S. B. Some spectral properties of a fourth-order boundary value problem with a spectral parameter in boundary conditions // Proceedings of the I International Scientific Conference of Young Scientists, vol. I, - Ganja: - 2016, - p. 341.
7. Guliyeva, S.B. Basis properties of the system of eigenfunctions of a fourth order eigenvalue problem with spectral parameter in the boundary conditions // – Baku: Transactions NAS Azerbaijan, ser. phys.–tech. math. sci., iss. math., – 2017. v. 37, no. 4, – p. 42–48.
8. Aliyev, Z.S., Guliyeva, S.B., Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load // Journal of Differential Equations, – 2017. v. 263, no. 9, – p. 5830–5845.
9. Aliyev, Z.S., Guliyeva, S.B., Spectral properties of a fourth order eigenvalue problem with spectral parameter in the boundary conditions // Filomat, – 2018. v. 32, no. 7, – p. 2421-2431.
10. Huseynov, O. R., Guliyeva, S. B. On the boundary value problem describing the bending vibrations of a rod, at one end of which a tracking force acts and an inertial load is concentrated \ \ Baku: Izvestiya Pedagogicheskogo Universiteta, series of Mathematical and Natural Sciences, - 2018. vol. 66, No. 2, - pp. 95-102.
11. Guliyeva, S.B. The properties of the eigenvalues and eigenfunctions of a vibration boundary value problem // – Baku: Caspian Journal of Applied mathematics, Ecology, and Economics, – 2019. v. 7, no. 1, – p. 25-31.
12. Gulieva, S. B. Spectral properties of a boundary value problem describing bending vibrations of a homogeneous rod, at one of the ends of which an inertial load is concentrated and a tracking force acts // Proceedings of the International Conference of the Voronezh Winter Mathematical School "Modern methods of the theory of functions and related problems", – Voronezh: Russia, – 2021, - pp. 97-98.

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