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ABSTRACT

of the dissertation for the degree of Doctor of Sciences

QUALITY PROPERTIES OF SOLUTIONS OF SECOND ORDER DEGENERATE EQUATIONS OF ELLIPTIC AND PARABOLIC TYPES

Specialty: 1211.01- Differential equations

Field of science: Mathematics

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Баку – 2021

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GENERAL CHARACTERISTIC OF THE WORK

Rationale of the theme and development degree.

The dissertation work is devoted to studies of quality properties of solutions of second order degenerate equations of elliptic and parabolic type. In the dissertation, the first boundary value problem for second order uniformly degenerating divergent elliptic equations and also Holder continuity of solutions, density of smooth functions in a weighted Sobolev space, Holder continuity and Harnack inequality for the solutions of p-Laplacian with Mackenhoupt weight degenerated into a part of the domain, Holder continuity and Harnack inequality for the solutions of an elliptic equation uniformly degenerated into parts of the domain of elliptic equation containing p Laplacian. Holder continuity and Harnack inequality for the solutions of uniformly degenerating into a part of the domain of elliptic (p,q) Laplacian are also studied. Furthermore, the Dirichlet problem for a class of nonuniformly degenerating elliptic and parabolic equations of second order is studied. The Holder continuity and Harnack inequality was studied for the solutions of nonuniformly degenerated parabolic equations.

Boundary value problems for degenerate elliptic equations are one of the important solutions of modern theory of partial differential equations.

The number published works on degenerate elliptic equations is very negligible. Concerning the second order degenerate elliptic equations, in this direction among first ones we can mention the paper of M.V.Keldysh (1951), where it is shown the case when the characteristical part of the boundary of the domain can become free from boundary conditions that can be replaced by the condition of boundedness of solutions. Later, in his paper A.V.Bitzadze showed that the boundedness condition can be replaced by a boundary condition with some weight function.

Degenerate elliptic equations arise in theory of small curves of rotation surfaces, in theory of shells, etc. Such equations play an important role in gas dynamics. These equations simulate the process related to diffusion and dissipation. The basic works on studying degenerate elliptic equations belong to Tricomi, Holmgren, Hellersdadt, Frankle, Jermen, Bitsadze, Babenko, Keldysh, Vishik, Kudryavtsev, Fiker, Vekua. Local regularity of weak solutions of degenerate linear elliptic equations in the divergent form were studied in the paper of Feibs, Kenig and Serapioni.

Researches of divergent nonuniformly elliptic equations with general structure weights get started with the papers of S.N.Kruzhkov and N.Trudinger.

Since at present there exist sufficiently developed theory of linear elliptic partial equations, there appears possibility of progress in theory of nonlinear equations. Significant successes in this direction was achieved for second order quasilinear elliptic equations owing to the papers of Schauder, Kaccioppoli, Lepe and others.

In nonstationary case this theory was developed in the papers of V.A.Solonnikov. These results were stated in detail in a number of books, for example in the book of K.Miranda, D.Hilbarg, N.Frudinger, Ch.Morri, etc. By the development of the tool of Sobolev space Holder continuity of the solutions of linear divergent parabolic equations of second order from which in its turn the estimations for the solution of elliptic equations were obtained, was proved by quite another methods. These results for elliptic equations were proved by the Yu Mozer by another methods.

Yu. Mozer's salient result was a proof for the solution of divergent elliptic equations of Harnack inequalities. Holder continuity of solutions and strong maximum principle are easily derived from the Harnack inequality. In 1964 Yu.Mozer took these results for a parabolic case. The Harnack inequality for weak solutions was generalized to divergent type quasilinear equations by Serrin and Turdinger.

One more proof of E.De.Georgie theorem in the case of elliptic equations was given by E.M.Landis.

Later it was revealed that generally speaking, these results are not extended to higher order equation and systems of equations and this was shown in 1968 independently by V.G.Mazya, E.D.George, E.Juiste and M.Mizanda. For nonlinear equations of higher order, these results were obtained in the papers of I.V.Skripnik.

However, the Holder inequality holds in the case of domain $n = 2l$ for $2l$ order elliptic equations and quasilinear analogs.

In the case of quasielliptic equations the similar result was obtained in the paper of R.V.Huseynov. The methods suggested by E.Di.George and Yu.Mozer were actively developed by E.Guisti, G.Stampakkia, O.A.Ladyzhenskaya, N.N.Uraltseva, J.Serrin, N.Trudinger, S.N.Kruzhkov, F.I.Mamedov and many other authors.

These ideas became applicable also to the solutions of divergent elliptic and parabolic equations regardless of their variational nature. Rather complete theory of linear and quasilinear elliptic equations with minor terms including elliptic p Laplacian type equations was constructed in the monograph of O.A.Ladyzhen-skaya and N.N.Uraltseva.

For linear parabolic equations (and nonlinear ones with the condition that order of coerciveness and growth of the principle part is linear) the similar theory may be found in the book of O.A.Ladyzhenskaya, N.N.Uraltseva and V.A.Solonnikov.

In 1982 there appeared a remarkable book E.Fabes, C.Kenig и R. Serapioni[1], devoted to regularity of solutions of divergent elliptic equations where the matrix $\{a_{ij}(x)\}$ satisfies the condition of uniform ellipticity and the weight $\omega(x)$ belongs to Mackenhoupt's A_2-class. In the same years there appears a paper by E.Fabes, D.S.Jerison, C.Kenig,[2] where for such equations the Wiener criterion of regularity of the boundary point was proved. This paper was followed by the papers of F.Chiarenza и M.Frasca in 1984, in

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[1] Fabes, E., Kenig, K., Serapioni, R. The local regularity of solutions of degenerate elliptic equations // Comm. Partial Differential Equations. -1982. v.7, -c. 77-116.

[2] Fabes, E.B. Jerison, D.S., Kenig, C.E. , The Wiener test for degenerate elliptic equations// Annals de l'institut Fourier. -1982, v.32, no 3. –c.151-182.

the papers of F.Chiarenza and R.Serapioni in 1984-1987 devoted to linear parabolic equations with weights from the Mackenhoupt classes.

The feature of equations with weight in the parabolic case is that here the spatial-time scale where the solution is considered, changes from the point to the point, respectively with the change of the values of weight.

Primarily, the methods used are similar to ones that are used in analysis on Euclidean space, the complexity is in obtaining necessary Friedrichs, Sobolev and Poincare type inequalities Yu.A.Alkhutov and V.V.Zhikov developed a technique for analyzing regularity of the solution of elliptic equations with a partial Mackenhoupt weight, i.e. in the situation when the domain is divided by a hyperplane into two parts and in each of parts the weight is a Mackenhoupt weight. The Holder continuity was obtained and it was shown that the Harnack inequality in the ordinary form, generally speaking is absent.

Object and subject of research. The main object of the thesis if a qualitative study of the properties of solutions to differential equations.

Goal and tasks of the research. Study of Holder continuity and Harnack inequality for the solutions of an elliptic equation containing p -Laplacian and uniformly degenerating by small parameter into a part of the domain. Study oif Holder continuity of the solutions of a linear degenerating elliptic equation involving the Lavrentyev effect. Absence of classic Harnak inequalities for p -Laplacian with a partial Mackenhoupt weight. Finding Harnack inequality corresponding to such an equation. To prove of unique solvability of the Dirichlet problem for linear divergent nonuniformly degenerating elliptic and parabolic equations of second order. To prove the Harnack inequality and Holder continuity of solution of such equations.

Research methods. In the paper, the methods of theory of differential equations, theory of functional spaces, theorems of imbedding and functional analysis were used. The main tool of the study is the developments of iteration technique a’la Moser-Di. George-Ladyzhenskaya-Uraltseva.

The main statements to be defended. The following main statements are defended:

1. Holder continuity of the solution of second order linear elliptic equations of divergent form with a weight on the plane in four-phase form when interphase bound are coordinate straightlines and the weight is a power function in each of phases.
2. Holder continuity of solutions and Harnack inequality for p-Laplace equation with partially Mackenhoupt weight in two-phase case when the interphase boundary is a hyperplane. .
3. Holder inequality and Holder continuity of solutions for a p-Laplace type equation uniformly degenerating by a small parameter into a part of the domain.
4. Holder continuity of solution of p-Laplacian with a Mackenhoupt weight uniformly degenerating by a small parameter into a part of the domain.
5. Holder continuity of solutions uniformly degenerating by a small parameter into a part of the domain of the $p(x)-$ Laplace type equation with a variable exponent p.
6. Harnack inequality and Holder continuity of solution of $p(x)-$Laplacian with two-phase piecewise constant exponent $p(x)$ in the case when a hyperphane is an interphase and in one of them the equation uniformly degenerates by a small parameter.
7. Estimation of the maximum of modulus of eigen functions of the Dirichlet problem for a second order divergent uniform elliptic operators containing a small parameter on a part of the domain.
8. Theorems on the existence and uniqueness of solutions of the Dirichlet problem for a class of second order, divergent form linear nonuniformly degenerating elliptic equations with minor terms.
9. Existence and uniqueness of solutions of the first boundary value problem for a class of linear divergent nonuniformly degenerating parabolic equations of second order, Harnack inequality and Holder continuity of solutions of such equations.

Scientific novelty of the study. In the dissertation work the following results were obtained:

1. Linear elliptic equations with a partial Mackenhoupt weight were studied. Holder continuity of solutions was proved.

2. p-Laplacian type nonlinear elliptic equations with a partial Mackenhoupt weight were considered. Holder continuity of solutions and Harnack inequality were proved.

3. $p(x)-$Laplacian type elliptic equations degenerating by a small parameter into a part of the domain were studied, Holder continuity of solutions was proved.

4. p-Laplacian type linear and nonlinear elliptic equations degenerating by a small parameter into a part of the domain were considered. Harnack inequality and Holder continuity of solutions of such weightless equations and equations with a Mackenhoupt weight were proved. p-Laplace equation with variable two-phase index p when the interphase is a hyperplane, was considered separately..

5. Linear nonuniformly degenerating elliptic equations were studied. Sovability of the Dirichlet problem for a class of the Dirichlet problem for a class of second order nonuniformly degenerating elliptic equations was proved.

6.The estimation of continuity of a boundary point of the solution of the Dirichlet problem for second order nonuniformly degenerating elliptic equations was given.

7. The estimation of the modulus of the first eigen function uniform by the parameter was found for a second order linear elliptic equation containing a large parameter on a part of the domain.

8.Weak solvability of the first boundary value problem in Sobolev weight spaces was proved for linear nonuniformly degenerating divergent parabolic equations.

9.The Harnack inequality for the solution of nonuniformly degenerating second order divergent parabolic equations was proved.

10.The Holder inequality of solutions of second order nonuniformly degenerating parabolic equations in the divergent form was shown.

Theoretical and practical value. The results obtained in the dissertation are new. The work is of theoretical character and fills a certain gap in theory of partial differential equations. Its results may be used for further development of quality properties of solutions for degenerating elliptic and parabolic equations.

Approbation and application. The basic results of the dissertation were discussed at the seminars of the departments «Mathematical physics equations» (corr. members of ANAS, prof. R.V.Huseynov), «Differential equations» (prof. A.B.Aliyev), «Nonharmonic analysis» (corr. members of ANAS, prof. B.T.Bilalov), at the institute seminar of IMM, at the seminar of the chair «Mathematical physics equations» BSU (acad. Yu.A.Mamedov), at the seminars of the chair «Differential and integral equations» BSU (prof. N.Sh.Iskenderov) and at the scientific seminar of the Institute of Applied Mathematics, BSU.

The main results of the dissertation were also reported at the seminar of VL.SU named after A.G. and N.G.Stoletov guided by V.V.Zhikov and Yu.A.Alkhutov and also at the following conferences: at the X International conference on mathematics and mechanics devoted to 45 years of IMM (Baku 2004), the All Union conference on mathematics and mechanics devoted to 50 years of corr. member of ANAS, prof. I.T.Mamedov (Baku 2005), «Theoretical and applied problems of operator equations» devoted to 75 years of prof. Yu.D.Mamedov (Baku 2006), at the International conference on mathematics and mechanics devoted to 70-th anniversary of acad. A.D.Gadjiev (Baku 2007), at the Republican conference «Actual problems of mathematics» devoted to 85 years of the national leader of Azerbaijan Heydar Aliyev (Baku 2008), at the III International conference devoted to 85 years of corr. member of RAS, prof. L.D.Kudryavtsev (2008), at the International conference on physical, mathematical and technical sciences (Nakhichevan 2008), at the International conference on mathematics and mechanics devoted to 50 years of IMM of ANAS (2009), in the materials of the international scientific conference devoted to 90 years of BSU, at the international conference «Spectral theory and its applications» devoted to 80-th anniversary of acad. F.G.Maksudov (2010), at the International conference devoted to acad. Z.Khalilov (2011), at the International conference «Theory of functions and problems of harmonic analysis» devoted to 100 years of acad. I.I.Ibrahimov (2012), at the conference devoted to 90-th anniversary of Heydar Aliyev (2013), at the International conference devoted to 55 years of IMM (Баку 2014), at the International conference on differential equations and dynamical systems (Suzdal 2010, 2016, 2018), Proceedings of the 6th international conference on control and optimization with industrial applications (Baku 2018).

The results of the dissertation were discussed with professors [I.T.Mamedov], [V.V.Zhikov], Yu.A.Alkhutov, [R.V.Huseynov], Yu.A.Mamedov, A.B.Aliyev, F.I.Mamedov, B.T.Bilalov, T.S.Gadjiev to whom I expressed my deep gratitude for discussions, valuable remarks and support.

The personal contribution of the author is in formulation of the goal and choice of research direction. Furthermore, all conclusions and obtained results and also research methods belong to the author.

Author's publications. Publications in the editions recommended by HAC under President of the Republic of Azerbaijan -24, conference materials- 1, abstracts of reports- 23.

The institution where the dissertation work was executed. The dissertation work was executed at the chair «Higher mathematics» of BSU.

Structure and volume of the dissertation (in signs, by indicating the volume of each structural subsection separately). The volume of the dissertation work – 489400 signs (title page – 288 signs, content – 2853 signs, introduction –81600 signs, chapter I – 134000 signs, chapter II – 136800 signs, chapter III – 42000 signs, chapter IV – 95000 signs). The list of used references 154 names.

MAIN CONTENT OF THE DISSERTATION

The dissertation consists of introduction, four chapters, references.

In introduction the urgency of the theme is justified, its degree of elaboration was shown, goal and tasks of the study was formulated, scientific novelty is given, theoretical and practical value of the study is noted, information on approbation of the work was represented.

Quality properties of the solutions of linear and nonlinear elliptic equations with a partial Mackenhoupt weight containing p-Laplacian are studied in chapter I. The main results of this chapter are in the author's papers [11,12,13,14,15, 18,35,42].

Section 1.1 studies solvability of a model uniformly degenerating second order elliptic equation for which the set of smooth functions is not dense in the appropriate weight Sobolev space W. The notion of W- and H- solutions is introduced, a unique solvability of appropriate W- and H- Dirichlet problems is proved.

Consider in a unique circle $B \subset R^2$ centered at the origin of coordinates a degenerating elliptic equation

$$Lu \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\omega(x)\frac{\partial u}{\partial x_i}\right) = 0\,, \tag{1}$$

with the weight

$$\omega(x) = \begin{cases} |x|^{-\alpha_1},\ x_1 > 0,\ x_2 > 0; \\ |x|^{\beta_1},\ x_1 < 0,\ x_2 > 0; \\ |x|^{-\alpha_2},\ x_1 < 0,\ x_2 < 0; \\ |x|^{\beta_2},\ x_1 > 0,\ x_2 < 0, \end{cases} \tag{2}$$

$0 < \alpha_i < 2,\ 0 < \beta_i < 2,\ i = 1{,}2.$

The following class of functions $W(B,\omega)$ is connected with equation (1):

$$W(B,\omega) = \{u : u \in W_1^1(B),\ (u^2 + |\nabla u|^2)\omega \in L_1(B)\} \tag{3}$$

Here $W_1^1(B)$ is a class of Sobolev space of functions summed in B together with generalized derivatives of first order. Below $W(B,\omega)$ is considered as a weight Sobolev space with the norm

$$\|u\|_W^2 = \int_B (u^2 + |\nabla u|^2)\omega dx\,.$$

Since $\omega^{-1} \in L_1(B)$, then the space $W(B,\omega)$ is complete. We will denote the closure of the sets of functions from $W(B,\omega)$ with a compact carrier in B by $W_0(B,\omega)$. The weight under consideration satisfies the Mackenhoupt A_2-condition in each fourth plane and the Friedrichs inequality holds

$$\int_B u^2 \omega dx \le C \int_B |\nabla u|^2 \omega dx \quad \forall u \in W_0(B,\omega). \tag{4}$$

Therefore, in the Sobolev space $W_0(B,\omega)$ we can give the norm by the equality

$$\|u\|_{W_0}^2 = \int_B |\nabla u|^2 \omega dx \ .$$

We introduce one of possible notions of the solution of equation (1).

Definition 1. The function $u \in W(B,\omega)$ is said to be the W solution of equation (1) if the integral identity

$$\int_B \nabla u \nabla \psi \omega dx = 0 \tag{5}$$

was fulfilled on test functions $\psi \in W_0(B,\omega)$.

The set of smooth functions in B is not dense in the spaces $W(B,\omega)$ and $W_0(B,\omega)$. The proof of this statement in the case when the weight function (2) satisfies the condition $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$, can be found in the papers of V.V.Zhikov.

In connection with what has been said, it has sense to determine the spaces $H(B,\omega)$ and $H_0(B,\omega)$ as a closure in $W(B,\omega)$ of the spaces $C^\infty(B) \cap W(B,\omega)$ and $C_0{}^\infty(B)$, respectively.

Definition 2. The function $u \in H(B,\omega)$ is said to be H-solution of equation (1), if integral set (5) is fulfilled on the test functions $\psi \in H_0(B,\omega)$.

The above W-solutions and H-solutions of equation (1) are connected with W-and H-Dirichlet problems

$$Lu_1 = 0 \ \text{ in } B, \quad u_1 \in W(B,\omega), \ h \in C^\infty(\overline{B}), (u_1 - h) \in W_0(B,\omega) \tag{6}$$

and

$Lu_2 = 0$ in B, $u_2 \in H(B,\omega)$, $h \in C^{\infty}(\overline{B})$, $(u_2 - h) \in H_0(B,\omega)$, (7)
respectively. For the weight function of the form

$$\omega(x) = \begin{cases} |x|^{-\alpha}, \ x_1 x_2 > 0; \\ |x|^{\alpha}, \ x_1 \, x_2 < 0, \end{cases} \tag{8}$$

the notion of W - and H -solutions of the equation of the considered type was introduced by V.V.Zhikov, a unique solvability of W -and H -Dirichlet problems and existence of their various solutions with one and the same boundary function was established in the papers of Yu.A.Alkhutov and V.V.Zhikov

Below $\omega(x)$ means a weight determined by equality (2). For the functions $u \in W(B,\omega)$ we use a polar system of coordinates centered at the origin of coordinates, given by the equality

$$u(x) = u(r,\theta),$$

and we use the denotation

$$D^{(1)} = B \cap \{x : x_1 > 0, \ x_2 > 0\}, \ D^{(2)} = B \cap \{x : x_1 < 0, \ x_2 < 0\},$$

$$D_r^{(i)} = D^{(i)} \cap \{x : |x| < r,\}, \ S_r^{(i)} = D^{(i)} \cap \{x : |x| = r\}, \ i = 1,2.$$

Let $d\mu_i = |x|^{-\alpha_i} dx, \ i = 1,2.$ For the measurable set $E \subset R^2$ and non-negative measurable function f we assume

$$\mu_i(E) = \int_E d\mu_i, \oint_E f(x) d\mu_i = \frac{1}{\mu_i(E)} \int_E f(x) d\mu_i, \quad i = 1,2.$$

It is shown that contraction of the function $u \in W(B,\omega)$ on $D^{(1)}$ and $D^{(2)}$ has the trades $u^{(i)}(0), \ i = 1,2$, at the origin of coordinates, equal to

$$u^{(i)}(0) = \lim_{R \to 0} \oint_{D_R^{(i)}} u(x) d\mu_i, \ i = 1,2, \tag{9}$$

and for any $R \in (0,1]$ it holds the Hardy inequality

$$\oint_{D_R^{(i)}} (u(x) - u^{(i)}(0))^2 |x|^{-2} d\mu_i \le C(a_i) \oint_{D_R^{(i)}} |\nabla u(x)|^2 d\mu_i, \ i = 1,2. \tag{10}$$

Below $W_2^1(D)$ means the Sobolev space of function, L_2-summable in the domain D together with generalized derivatives of first order. We have the following statement.

Lemma 1. For belonging the function $u(x)$ to the weight Sobolev space $H(B,\omega)$ it is necessary and sufficient the equality $u^{(1)}(0)=u^{(2)}(0)$ to be fulfilled. This time the space H has a co-dimension 1 in W. The function $u \in W(B,\omega)$ given below:

$$u(x)=\begin{cases} 1, & for \ \ x_1>0, x_2>0; \\ \sin\theta, & for \ \ x_1<0, x_2>0; \\ 0, & for \ \ x_1<0, x_2<0; \\ \cos\theta, & for \ \ x_1>0, x_2<0 \end{cases}$$

does not belong to the Sobolev space $H(B,\omega)$.

The spaces $W_0(B,\omega)$ and $H_0(B,\omega)$ are Hilbert spaces with the scalar product

$$\langle u,\upsilon\rangle=\int_B \nabla u \nabla \upsilon \omega dx .$$

We have the following theorem.

Theorem 1. Problems (6)-(7) are uniquely solvable and there exists a boundary function $h \in C^\infty(\overline{B})$, for which the solutions $u_1(x)$ and $u_2(x)$ are different.

In section 1.2 Holder continuity of W- and H- solutions is studied.

Theorem 2. If the weight $\omega(x)$ satisfies condition (2), then H - solutions of equation (4) are Holder in B, while W solutions that are not H -solutions are discontinuous at the origin of coordinates and are Holder in $B \cap \{x: x_1 \geq 0,\ x_2 \geq 0\}$ and $B \cap \{x: x_1 \leq 0,\ x_2 \leq 0\}$.

Since the weight function $\omega_i(x)$ satisfies the A_2-Mackenhoupt condition and one even with respect to coordinate straightlines, then the following Sobolev inequality is valid

$$\left(\oint_{D_r^{(i)}} |\varphi|^4 d\mu_j \right)^{\frac{1}{2}} \le Cr^2 \oint_{D_r^{(i)}} |\nabla \varphi|^2 d\mu_j, \;\; \varphi \in W_0\,(B,\omega),\;\; j = 1,23,4\,. \quad (11)$$

From (11) and Moser's iteration technique we arrive at the following statement.

Lemma 2. Any W - solution and H - solution of equation (1) are locally bounded in B.

The key role in the proof of the theorem is played by the weight estimation of the solutions of the considered equation. To formulate it, we denote by B_r, where $r \le R$, open circles of rather small radius r centered at the churches $S_R^{(i)}, i = 1,3,$ assuming $D_r^{(i)} = B_r \cap D^{(i)}$. Assume

$$\vartheta(x) = |u(x)|^{1+\varepsilon} + R^{\nu}\,,\;\; \nu > 0\,,$$

where u is the solution of equation (1). The introduced function is a bounded positive subsolution of the same equation. Below we assume

$$\nu_0 = \min\left\{ \frac{\beta_2 + \alpha_1}{2},\; \frac{\beta_2 + \alpha_2}{2} \right\} \;\text{ and }\; 0 < \nu < \nu_0\,.$$

Theorem 3. There exists positive constants C_1 *and* C_2, *independent of* ϑ, ν *and* r *such that subject to the condition*

$$1 \le \gamma \le C_1 R^{\nu - \nu_0}$$

for any $0 < \rho < r$ *the following inequalities are valid*

$$\left(\oint_{D_\rho^{(1)}} \vartheta^{2(\gamma+1)} d\mu_1 \right)^{1/2} \le C_2 (\gamma+1)^2 \left(\frac{r}{r-\rho} \right)^2 \oint_{D_\rho^{(1)}} \vartheta^{\gamma+1} d\mu_1,$$

$$\left(\oint_{D_r^{(3)}} \vartheta^{(\gamma+1)} d\mu_3 \right)^{1/2} \le C_2 (\gamma+1)^2 \left(\frac{r}{r-\rho} \right)^2 \oint_{D_r^{(3)}} \vartheta^{\gamma+1} d\mu_3\,.$$

From the results of Yu.A.Alkhutov and V.V.Zhikov's papers on Holder continuity of the solution of elliptic equations with partially weight it follows that the W - solutions and H -solutions of equation (1) are Holder in $B\setminus\{(0,0)\}$. Therefore, the proof of theorem 2 is reduced to studying the behavior of solutions at the origin of coordinates. Modification of Moser's method develop by the another and based on the estimations of theorem 3 and statement of lemma 1 is the base of the proof.

After theorem 3 the reasonings are devoted to the proof of the Moser estimation for the maximum of the subsolution $\vartheta(x)$ on the archs $S_R^{(i)}$, where $i=1,3,$ through the mean values of the integrals along the sets $D_R^{(i)}$. For that it is necessary to integrate the inequalities obtained in the given theorem. However, the ordinary scheme of proof can not be used here because of restrictions on the constant γ. The suggested method is in integration of inequalities along the sequence of circles with geometrically decreasing sequence of radii and satisfying the restrictions on the constant γ, allows for a finite number of step to pass to the circle $B_r^{x_0}\subset D^{(i)}$. After this, ordinary reasonings based on the Moser technique are used.

In section 1.3 necessary and sufficient conditions on a special kind weight providing density of smooth function in the Sobolev weight space are found.

Let us consider on a unit circle $B=\{x:|x|<1\}$ of the Euclidean plane R^2 the Sobolev weight space with the weight

$$\omega(x)=\begin{cases} f^{-1}(|x|), & for \quad x_1\, x_2>0 \\ f(|x|), & for \quad x_1\, x_2<0. \end{cases}$$

Fulfilment of the following conditions is required from the function f, participating in determination of the weight. We will assume that $f(t)$ is continuous, does not decrease on $(0,1]$ and

$$f(2t)\le c\, f(t) \quad \forall t\in\left(0,\frac{1}{2}\right], \tag{12}$$

$$\sup_{t\in(0,1)}\left(\int_t^1\frac{f(\tau)}{\tau}d\tau\right)\left(\int_0^t\frac{d\tau}{f(\tau)}\right)<\infty. \tag{13}$$

After this, not especially stipulating, we will additionally assume $\omega^{-1} \in L_1(B)$. Hence it follows that the space $W(B,\omega)$ is complete. In particular, the functions of the form $f(t) = \ln^{\gamma} \frac{1}{2t} \cdot t^{-\alpha}$, where $\gamma \in (-\infty,+\infty)$, $\alpha \in (0,2)$ satisfy the given conditions.

In the case when $f(t) = t^{-\alpha}$, $\alpha \in (0,2)$, this issue was solved in V.V.Zhikov's papers. In our case, subject to conditions (12),(13) the set of functions from $C^{\infty}(\overline{B}) \cap W(B,\omega)$, generally speaking are not dense in $W(B,\omega)$. In this connection, we determine the space $H(B,\omega)$ as a closure in $W(B,\omega)$ of the set $C^{\infty}(\overline{B}) \cap W(B,\omega)$.

Theorem 4. The function $u \in W(B,\omega)$ belongs to the space $H(B,\omega)$ if and only if

$$L_1(u) = L_3(u),$$

where

$$\lim_{r\to 0} \frac{2}{\pi} \int_0^{\pi/2} u(r,\theta)\, d\theta = L_1(u)\,, L_3(u) = \lim_{r\to 0} \frac{2}{\pi} \int_{\pi}^{3\pi/2} u(r,\theta) d\theta,$$

Section 1.4 was devoted to the interior a priori estimation of the Holder norm of weak solutions of second order uniformly degenerating quasilinear elliptic equations of divergent form.

Consider in the domain $D \subset R^n$, $n \geq 2$, the elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\omega(x) |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) = 0, \ p = const > 1, \qquad (14)$$

where $\omega(x) \geq 0$. In order to determine the solutions, we introduce a class of functions

$$W_{loc}(D,\omega) = \{u : u \in W_{loc}^{1,1}(D), |\nabla u|^p \omega \in L_{loc}^1(D)\}$$

where $W^{1,1}(D)$-is a class Sobolev space. Under the solution of equation (14) we understand the function $u \in W_{loc}(D,\omega)$, for which the integral set

$$\sum_{i=1}^{n} \int_D \omega\,(x)|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial \xi}{\partial x_i} dx = 0$$

was fulfilled on finite test functions $\xi \in W_{loc}(D,\omega)$. The goal is to prove the Holder continuity of the solution of equation (14). There is a great number of papers devoted to this topic. The case when the weight function $\omega(x)$ satisfies the Mackenhoupt A_p-condition was most studied. The case $p \neq 2$ was studied in the papers of Heinonen J., Kilpelainen T., Martio O[3]. Remind that the weight $\omega(x)$, determined in all the space R^n, satisfies the A_p-condition if

$$\sup\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}(x)dx\right)^{p-1} < \infty,\ 1 < p < \infty,$$

where the supremum is taken on all spheres $B \subset R^n$.

The standard example of such a weight is a power function $\omega(x) = |x|^{\alpha}$, where $-n < \alpha < n(p-1)$, and also $\omega(x) = |x_n|^{\alpha}$, where $-1 < \alpha < p-1$.

The important corollaries of Mackenhoupt's A_p-condition are the doubling conditions

$$\omega(B_{2r}) \leq c\omega(B_r), \tag{15}$$

the Sobolev inequalities

$$\left(\oint_{B_r} |\varphi|^{pk} d\mu\right)^{\frac{1}{k}} \leq c(n,p)r^p \oint_{B_r} |\nabla\varphi|^p d\mu,\ \varphi \in C_0^{\infty}(B_r),\ k = \frac{n}{n-1}, \tag{16}$$

the Friedrichs inequalities

$$\int_{B_r} |\varphi|^p d\mu \leq c(n,\nu,p)r^p \int_{B_r} |\nabla\varphi|^p d\mu,$$

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[3] Heinonen, J., Kilpelainen, T., Martio, O. Nonlinear potential theory of degenerate elliptic equations// Mineola. NY: Dover Publ. Inc.,- 2006. Xii+404 pp.

$$\varphi \in C^{\infty}(\overline{B}_r),\ \varphi\big|_E = 0,\ |E| \ge \gamma |B_r|,\ \gamma > 0\,.$$

In the Yu.A.Alkhutov and V.V.Zhikov's paper weight functions of a more general form were considered. More exactly, it is assumed that the hyperplane $\Sigma = \{x : x_n = 0\}$ divides the domain D into two subdomains

$D^{(1)} = D \cap \{x : x_n > 0\}$ and $D^{(2)} = D \cap \{x : x_n < 0\}$ and

$$\omega(x) = \begin{cases} \omega_1(x) \ \ в \ \ D^{(1)}, \\ \omega_2(x) \ \ в \ \ D^{(2)}, \end{cases} \tag{17}$$

where each of even with respect to Σ weight functions $\omega_i(x),\ i = 1,2,$ satisfies the Mackenhoupt A_p-condition. Furthermore, for the spheres B_r centered at Σ for almost all $x \in B_r$ for $r \le r_0$ the following inequality

$$\frac{\omega_1(x)}{\omega_1(B_r)} \le c \frac{\omega_2(x)}{\omega_2(B_r)} \tag{18}$$

with the constant c, independent of r and x was fulfilled. In particular, in r_0 vicinity Σ

$$\omega_1(x) \le c\omega_2(x)\,.$$

For these weights, the doubling condition (15) and Sobolev's inequality (16) violate in the general case. In Yu.A.Alkhutov and V.V.Zhikov's paper it was shown that for $p = 2$ subject to conditions (17) and (18) the solutions of equations (14) are Holder continuous and in this time the classic Harnack inequality is absent in the general case.

The key role in the proof of the Holder continuity of solutions is played by the following statement.

Lemma 3. *Let $\vartheta(x)$ be a positive bounded subsolution of equation (14), i.e.*

$$\int_D \sum_{i=1}^{n} \omega(x) |\nabla \vartheta|^{p-2} \frac{\partial \vartheta}{\partial x_i} \frac{\partial \xi}{\partial x_i} dx \le 0,\ \forall \xi \in C_0^{\infty}(D),\ \xi \ge 0 \text{ in } D\,.$$

Then for any sphere $B_{8R} \subset D$ the following inequality is fulfilled

$$\sup_{B_R} \vartheta(x) \le c\left(\oint_{B_{2R}^{(1)}} \vartheta^p(x)d\mu_1 + \oint_{B_{2R}^{(2)}} \vartheta^p(x)d\mu_2 \right)^{1/p},$$

where the constant c *depends only on* n, p, u, ω.

The domain $D^{(1)}$ and $D^{(2)}$ play various roles in the proof of this lemma. This is connected with the fact that in the spheres with a center dividing the hyperplane, there is no a weight Sobolev imbedding theorem with increased summability index.

Then we prove the following main result.

Theorem 5. If ω satisfies conditions (17) and (18), where ω_1 and ω_2 are even with respect to the hyperplane $\sum$ functions, belonging to the Mackenhoupt A_p class, then all the solutions of equation (14) are Holder in D.

In section 1.5 the absence of the classic Harnack inequality for the solutions of equation (14) with the weight satisfying conditions (17), (18) is proved and the Harnack inequality corresponding to this equation is established.

Earlier it was shown that if $\omega \in A_p$, then the solution of the equation (14) are Holder in D and for all non-negative in $B_{4R} \subset D$ solutions, the Harnack classic inequality is proved

$$\inf_{B_R} u \ge const \cdot \sup_{B_R} u. \tag{19}$$

We have established that if in the spheres B_R centered at $\sum \cap D$ the condition

$$\frac{\omega_2(B_r)}{\omega_1(B_r)} \to \infty \text{ for } r \to 0,$$

is fulfilled, then the classic Harnack inequality (19) and Sobolev inequality (17) do not hold knowingly The proof of this result is based on the estimations of volume potential.

Since the classic Harnack inequality (19) violetes in the spheres centered at the hyperplane $\sum$, in the formulation of the result just such spheres participate and below we assume

$$B_R^{-} = B_R \cap \{x : -R < x_n < -R/2\}. \quad (20)$$

The following statement holds.

Theorem 6. *If the weight* $\omega(x)$ *satisfies conditions (17), (18) and* $u(x)$ *is a non-negative solution of equation (14) in the sphere* $B_{4R} \subset D$ *centered at* $\sum$*, then we have the following inequality*

$$\inf_{B_R} u \geq \gamma \sup_{B_R^{-}} u, \quad (21)$$

where the positive constant $\gamma < 1$ *is independent of* u *and* R.

The Holder property of the solution at the points $\sum \cap D$ and as a corollary, Holder continuity of solutions on all the domain D follows from theorem 6.

In section 1.6 we research $p(x)-$Laplace type equation with variable exponent p, uniformly degenerating by a small parameter into a part of the domain. More exactly in the domain $D \subset R^n, n \geq 2$, we consider the class of elliptic equations

$$L_\varepsilon u = div\left(\omega_\varepsilon(x)|\nabla u|^{p(x)-2}\nabla u\right) = 0 \quad (22)$$

with measurable exponent $p(x)$ such that

$$1 < p_1 \leq p(x) \leq p_2 \quad \text{almost everywhere in } D, \quad (23)$$

and with a positive weight $\omega_\varepsilon(x)$, that will be determined now. It is assumed that the domain D was divided by the hyperplane $\sum = \{x : x_n = 0\}$ into the parts $D^{(1)} = D \cap \{x : x_n > 0\}$, $D^{(2)} = D \cap \{x : x_n < 0\}$ and

$$\omega_\varepsilon(x) = \begin{cases} \varepsilon, & if \quad x \in D^{(1)} \\ 1, & if \quad x \in D^{(2)} \end{cases}, \ \varepsilon \in (0,1]. \quad (24)$$

In order to determine the solutions of equation (22) we introduce the class of functions

$$W_{loc}(D) = \left\{u : \ u \in W_{loc}^{1,1}(D), |\nabla u|^{p(x)} \in L_{loc}^1(D)\right\},$$

where $W_{loc}^{1,1}(D)$ is a Sobolev class of functions in D together with generalized derivatives of first order.

Under the solution of equation (22) we understand the function $u \in W_{loc}(D)$, satisfying the integral identity

$$\int_D \omega_\varepsilon(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla \psi dx = 0 \tag{25}$$

on the trial functions $\psi \in C_0^\infty(D)$.

Density of smooth functions in the introduced class of solutions $W_{loc}(D)$ is of importance. In the paper of V.V.Zhikov and Kh.Fan it was shown that if the logarithmic condition

$$|p(x)-p(y)| \le \frac{c}{\ln \frac{1}{|x-y|}} \text{ при } x, y \in D, \quad |x-y| < \frac{1}{2} \tag{26}$$

is fulfilled, then for the arbitrary function $u \in W_{loc}(D)$ there exists a sequence $\{u_j\}$, where $u_j \in C^\infty(D)$, is such that in the arbitrary subdomain $\overline{D}' \subset D$ the following relation is fulfilled

$$\lim_{j\to\infty}\|u_j - u\|_{W^{1,1}(D')} = 0, \ \lim_{j\to\infty}\int_{D'}|\nabla u_j|^{p(x)}dx = \int_{D'}|\nabla u|^{p(x)}dx.$$

At present the condition (26) plays a great role in theory of Sobolev spaces with variable summability index and in Holder continuity of $p(x)$-harmonic functions. In his paper Yu.A.Alkhutov proves a priori estimation of the Holder norm of solutions of equation (22) for condition (26) in the case when $\varepsilon = 1$. In the present paper we assume that for $i = 1,2$

$$|p(x)-p(y)| \le \frac{c_0}{\ln \frac{1}{|x-y|}} \qquad npu \ x, y \in D^{(i)}, |x-y| < 1/2, \tag{27}$$

in some vicinity of $D \cap \sum$, we have the inequality

$$p(x) \ge p(\tilde{x}) \qquad for \quad x \in D^{(2)}, \tag{28}$$

where $\tilde{x}$ is a point symmetric to x with respect to the hyperplane $\sum$. In the paper of Yu.A.Alkhutov it is shown that subject to the condition (27) the set of smooth functions is dense in the class

$W_{loc}(D)$. Hence it follows that in the integral identity (25) one can use trial functions $\psi \in W_{loc}(D)$ with a compact support in D. We are interested in independence of the Holder exponent α on a small parameter ε.

Let $\{u^{\varepsilon}(x)\}$- be a family of solutions of equation $L_{\varepsilon}u^{\varepsilon} = 0$, bounded in L_{∞} uniformly with respect to ε on compact subsets D. We prove the following statement.

Theorem 7. If conditions (23), (24), (27) and (28), are fulfilled, then there exists a constant $\alpha \in (0,1)$, independent of ε, such that the family $\{u^{\varepsilon}(x)\}$ is compact in $C^{\alpha}(D')$ in the arbitrary subdomain $\overline{D}' \subset D$.

The second part of the dissertation was devoted to the proof of Holder continuity of solutions and Harnack inequality for second order elliptic divergent equations uniformly convergent in a small parameter into a part of the domain. The basic results of this chapter are in the author's paper [20, 21, 24, 34,36,37,38,39,40,42,45,46,47].

In section 2.1 in the domain $D \subset R^{n}$, $n \geq 2$, we consider a family of linear elliptic equations

$$L_{\varepsilon}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\omega_{\varepsilon}(x)\frac{\partial u}{\partial x_j} \right) = 0 \qquad (29)$$

assuming that the domain D is divided by the hyperplane $\sum = \{x : x_n = 0\}$ into the parts $D^{(1)} = D \cap \{x : x_n > 0\}$ and $D^{(2)} =$ $= D \cap \{x : x_n < 0\}$ with the weight $\omega_{\varepsilon}(x)$, satisfying condition (24), and with a measurable symmetric matrix satisfying the condition of uniform ellipticity $\{a_{ij}(x)\}$,

$$\gamma^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \gamma|\xi|^2. \qquad (30)$$

Under the solution of equation (29) we understand a function $u \in W_{2,loc}^{1}(D)$, satisfying the integral identity

$$\sum_{i,j=1}^{n} \int_D a_{ij}(x)\omega_\varepsilon(x)u_{x_i}\varphi_{x_j}\,dx = 0$$

on trial functions $\varphi \in \overset{\circ}{W}{}_2^1(D)$ with a compact carrier in D.

Let $\{u^\varepsilon(x)\}$ be a family of solutions of the equation $L_\varepsilon u^\varepsilon = 0,$ bounded in L_∞ uniformly with respect to ε on compact subsets D.

We get the following result.

Theorem 8. There exists a constant $\alpha \in (0,1),$ dependent only on the dimension of the space n and the constant γ from condition (30) and such that the family $\{u^\varepsilon(x)\}$ is compact in $C^\alpha(D')$ in arbitrary subdomain $\overline{D'} \subset D$.

Section 2.2 was devoted to generalization of theorem 8 for the case of p-Laplacian type equation degenerating in a small parameter ε into a part of the domain. In the domain $D \subset R^n, n \geq 2,$ divided by the hyperplane into two parts, we consider a family of elliptic equations

$$L_\varepsilon u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\omega_\varepsilon(x)\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = 0, \quad p > 1 \tag{31}$$

with the weight $\omega_\varepsilon(x)$, satisfying condition (24).

Under the solution of equation (31) we understand a function $u \in W_{p,loc}^1(D)$, satisfying the integral identity

$$\sum_{i=1}^{n} \int_D \omega_\varepsilon(x)\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx = 0$$

on the trial functions $\varphi \in \overset{\circ}{W}{}_p^1(D)$ with a compact carrier in D.

It is known that for each fixed value $\varepsilon \in (0,1]$ any solutions of equation (31) in the arbitrary subdomain $\overline{D'} \subset D$ belongs to the space $C^\alpha(D')$ of functions Holder in D'. We are interested in independence of the exponent α of ε .

We consider the family of solutions of the equation $\{u^{\varepsilon}(x)\}$ bounded in $L_{\varepsilon}u^{\varepsilon}=0$, uniformly with L_{∞} respect to ε on compact subsets D. The base of this section is to prove the following statement.

Theorem 9. There exists a constant $\alpha \in (0,1)$, dependent only on the dimension of the space n and p such that the space $\{u^{\varepsilon}(x)\}$ is compact in $C^{\alpha}(D')$ in arbitrary subdomain $\overline{D'} \subset D$.

In section 2.3 we prove the analog of the Harnack inequality for non-negative solutions of equation (31).

If $\omega_{\varepsilon} \equiv 1$, then for a non-negative solution of equation (31) in the ball $B_{4R} \subset D$ we have the classic Harnack inequality (19).

We are interested in the analog of the Harnack inequality for non-negative solutions with a constant independent of ε. We established that the Harnack inequality (19) in the balls centered at the hyperplane with a constant independent of ε, does not hold. The main goal is to obtain the analog of the Harnack inequality centered at $\sum$. Below we use the denotation from (20).

Theorem 10. If $u(x)$ is a non-negative solution of equation (31) in the ball $B_{4R} \subset D$ centered at $\sum$, then we have the inequality (21) with a constant independent of u, R, ε.

We statement of lemma 8 follows from inequality (21).

In section 2.4 we also study an equation of the form (31) in the domain $D \subset R^{n}$, $n \geq 2$, in the domain D, divided by a hyperplane into two parts with the weight

$$\omega_{\varepsilon}(x) = \begin{cases} \varepsilon\omega(x), \ x \in D^{(1)}, \\ \omega(x), \ x \in D^{(2)}, \varepsilon \in (0,1], \end{cases} \tag{32}$$

where $\omega(x)$ is a weight satisfying the Mackenhoupt A_{p}-condition.

Furthermore, is assumed that in the open balls B_{R_0} of rather small radius R_0 centered at the hyperplane $\sum$ for almost all points

x from the semi-sphere $B_{R_0} \cap \{x : x_n > 0\}$ the following inequality is fulfilled:

$$\omega(x) \le \gamma\omega(x'), \ \gamma = const > 0, \tag{33}$$

where x' is a point symmetric to x with respect to the hyperplane $\sum$. In particular, the weights $|x|^{\alpha}$, where $-n < \alpha < n(p-1)$, and $|x_n|^{\alpha}$, where $-1 < \alpha < p-1$ satisfy this condition. Furthermore, every weight satisfying the Mackenhoupt A_p-condition and is even with respect to the hyperplane $\sum$ is appropriate. It is well known that for fixed value $\varepsilon \in (0,1]$ any solution of the equation under consideration in an arbitrary subdomain $\overline{D}' \subset D$ belongs to the space $C^{\alpha}(D')$ functions Holder in D' We are interested in independence of the exponent α at ε. Below, as earlier, $\{u^{\varepsilon}(x)\}$ - is a family of solutions of equations $L_{\varepsilon}u^{\varepsilon} = 0$, bounded in L_{∞} uniformly with respect to ε on compact subsets D.

Theorem 11. If the weight ω satisfies the Mackenhoupt A_p-condition and condition (33) is fulfilled, then there exists a constant $\alpha \in (0,1)$, dependent only in p, dimension of the space n, constant γ from (33) and the weight ω, such that the family $\{u^{\varepsilon}(x)\}$ is compact in $C^{\alpha}(D')$ of arbitrary subdomain $\overline{D}' \subset D$.

In section 2.5 we continue to study equation (31) with the weight $\omega_{\varepsilon}(x)$, satisfying the conditions (32) and (33).

Here we give no classic Harnack inequality (19) in the spheres centered at the hyperplane with a constant independent of ε.

The main result where we use the denotation (20), is in the following statement.

Lemma 4. If the weight satisfies the Mackenhoupt A_p-condition and supposition (33) is fulfilled, then for any $q > 0$ we have the following inequality

$$\inf_{B_R} u(x) \geq C\left(\oint_{B_{2R}} v^{-q}(x)d\mu \right)^{-1/q}$$

with the constant C, independent of u and R.

Lemma 5. If the weight satisfies the Mackenhoupt A_p-condition and supposition (33) is fulfilled, then for any sphere $B_{2r} \subset B_{3R}$ – we have the estimation

$$\int_{B_r} \left|\nabla \ln v\right|^p d\mu \leq Cr^{-p}\omega(B_r),$$

where the constant C is independent of u, r, R and ε.

Theorem 12. If the weight $\omega(x)$ satisfies the Mackenhoupt A_p - condition and supposition (33) is fulfilled, then for nonnegative in the sphere $B_{4R} \subset D$ centered at $\sum$, solution u of equation (31) it is valid inequality (21), where the positive constant is independent of $u, R\, u\, \varepsilon$.

In section 2.6 in the domain $D \subset R^n$, $n \geq 2$, divided by the hyperplane into two parts, we consider a family of elliptic equations (22) with a weight from (24) and exponent $p(x)$ of the form

$$p(x) = \begin{cases} q, \, if \;\; x \in D^{(1)} \\ p, \, if \;\; x \in D^{(2)}, \end{cases} \qquad 1 < q < p. \tag{34}$$

For determining the solution we introduce a class of functions related to the exponent $p(x)$:

$$W_{loc}(D) = \{u : u \in W_{loc}^{1,1}(D), |\nabla u|^{p(x)} \in L_{loc}^1(D)\},$$

where $W_{loc}^{1,1}(D)$ is a Sobolev space of functions locally summable in D together with first order generalied derivatives.

Under the solution of equation (22) we understand a function $u \in W_{loc}(D)$, satisfying the integral identity

$$\int_D \omega_\varepsilon(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx = 0$$

on trial functions $\varphi \in C_0^\infty(D)$.

Again, as earlier, $\{u^{\varepsilon}(x)\}$ means a family of solutions of equations $L_{\varepsilon}u^{\varepsilon}=0,$ bounded in L_{∞} uniformly with respect to ε on compact subsets D.

We prove the following statement.

Theorem 13. There exists a constant $\alpha\in(0,1)$, dependent only on the dimension of space n, u and constants p, q from condition (34), such that the family $\{u^{\varepsilon}(x)\}$ is compact in arbitrary subdomain $\overline{D}'\subset D.$

The proof of this result is based on two auxiliary results where in $B_R\subset D$ are the spheres centered at the hyperplane Σ, $M=\sup\limits_{B_{R_0}}|u(x)|$, where, $R_0\le 1/4$, and for $R\le R_0/6$ we assume

$$M_6=\sup_{B_{6R}}u,\quad m_6=\inf_{B_{6R}}u,\quad \vartheta(x)=\ln\frac{M_6-m_6+2R}{M_6-u(x)+R}.$$

Lemma 6. For any $R\le\rho<r\le 3R$ we have the inequality

$$\sup_{B_\rho}\vartheta\le C(n,p,q,M)\left(\frac{r}{r-\rho}\right)^{a}\left(\fint_{B_r}\vartheta^{p}\,dx\right)^{1/p}$$

with a constant $a(n,p)>0$.

From the above lemma, by means of the Moser method we establish the following fact.

Lemma 7. We have the estimation

$$\sup_{B_R}\vartheta\le C(n,p,q,M)\fint_{B_{2R}}\vartheta dx.$$

In the course of the proof of the formulated theorem it was established that any solution of equation (31) is Holder continuous in the domain D with an exponent dependent only on n, p and q. Note that for every fixed value $\varepsilon\in(0,1]$ the Holder continuity of solutions follows from the results of the paper of E.Acerbi и N.Fusco[4].

In section 2.7 we prove the Harnack inequality for non-negative solutions of (p,q)-Laplacian.

Theorem 14. *If conditions (24),(34) are fulfilled and is a non-negative solution of equation (22) in the space* $B_{8R} \subset D$ *centered at the hyperplane* $\sum$*, then in the concentrically sphere* B_R *of radius* R *the following inequality is valid*

$$\inf_{B_R} u + R \geq C(n,p,q) \sup_{B_R^{-}} u$$

where in the denotation (20) is used, the constant C depends only on n,p,q.

The proof of the theorem 14 is based on the following statement where we assume

$$\vartheta(x) = \begin{cases} w(x), & \text{если} \quad x \in D^{(2)} \\ \min(w(x), \tilde{w}(x)), & \text{если} \quad x \in D^{(1)}. \end{cases}$$

Lemma 8. For any $q_0 > 0$ the following inequality is valid

$$\inf_{B_R} \vartheta(x) \geq C(n,p,q,q_0) \left(\oint_{B_{2R}} \vartheta^{-q_0}(x)dx \right)^{-1/q_0}.$$

or, as $w \geq \vartheta$, then

$$\inf_{B_R} w(x) \geq C(n,p,q,q_0) \left(\oint_{B_{2R}} \vartheta^{-q_0}(x)dx \right)^{-1/q_0}.$$

In chapter III we consider second order nonuniformly degenerating divergent elliptic equations. The basic results of this chapter are in the author's papers [2,3,7,9, 23,41].

In section 3.1 we prove a unique weak solvability in anisotropic Sobolev spaces of the first boundary value problem in

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[4] Acerbi, E., Fusco, N. A transmission problem in the calculus of variations // Calc. Var. Partial Differ. Equ. -1991. v.2, no. 1. -p.1-16.

the bounded domain D of diameter d for a second order nonuniformly degenerating elliptic equations of the form

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u =$$

$$= f(x) + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \quad u/\partial D = 0, \qquad (35)$$

where $f \in L_2(D),\ \frac{f_i}{\sqrt{\lambda_i}} \in L_2(D);\ i = 1,\ldots,n.$

The matrix of higher coefficients is measurable, symmetric and for all $\xi \in E_n,\ x \in D$

$\gamma \sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x)\xi_i^2,\ \gamma \in (0,1]$ is a constant, (36)

where

$$\lambda_i(x) = \left(|x|_\alpha\right)^{\alpha_i},\ |x|_\alpha = \sum_{i=1}^{n} |x_i|^{\frac{2}{2+\alpha_i}},\ \alpha_i \ge 0;\ i = 1,\ldots,n. \qquad (37)$$

In this section we prove the following Sobolev type inequality. Below by $W_{p,\alpha}^1(D)$ we denote a Banach space of functions $u(x)$ given on D, with the finite norm

$$\|u\|_{W_{p,\alpha}^1(D)} = \left[\int_D \left(|u|^p + \sum_{i=1}^{n} \left(\lambda_i(x)\right)^{p/2} \left| \frac{\partial u}{\partial x_i} \right|^p \right) dx \right]^{1/p},\ (1 < p < \infty),$$

and by $\mathring{W}_{p,\alpha}^1(D)$-the subspace $W_{p,\alpha}^1(D)$, the dense subset in which is the totality of all functions from $C_0^\infty(D)$. Assume $\alpha^- = \min\{\alpha_1,\ldots,\alpha_n\},\ \alpha^+ = \max\{\alpha_1,\ldots,\alpha_n\}.$

Theorem 15. *Let the funtions* $\lambda_i, i=1,\ldots,n$ *be determined by equality* (37). *Then for any* $p, \frac{2n}{n+2}<p<2$ *and any function* $u\in \overset{\circ}{W}{}^1_{2,\alpha}(D)$ *subject to the condition*

$$\alpha^+ < \frac{4-2p}{3p-2} \tag{38}$$

the following estimation is valid

$$\|u\|_{L_{\frac{np}{n-p}}(D)} \le c_2\left(\int_D \sum_{i=1}^n \lambda_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2 dx\right)^{1/2}, \tag{39}$$

$$\textit{where } c_2 = c_1\left(2d^{n-1}\right)^{\frac{2-p}{2p}}\left(\sum_{i=1}^n \gamma_i^{\frac{p-2}{p}}\left(\frac{d}{2}\right)^{\frac{\gamma_i(2-p)}{p}}\right)^{1/2},$$

$$\gamma_i = \frac{4-2p-\alpha_1(3p-2)}{(2+\alpha_i)(2-p)},\ i=1,\ldots.,n.$$

We will assume that for the minor coefficients of the operator L the following conditions are fulfilled

$$\omega = \sum_{i=1}^n \left\|\frac{b_i}{\sqrt{\lambda_i}}\right\|_{L_r(D)} < \frac{\mu}{c_3}, \tag{40}$$

$$c(x)\in L_r(D),\ c(x)\le 0. \tag{41}$$

Here $r>n$, the constant c_3 is determined in the same way as the constant c_2 for $p=\frac{2rn}{(n-1)r+n}$. By means of the formulated imbedding theorem we establish a unique weak solvability of the first boundary value problem.

Theorem 16. If conditions (36)-(41), *are fulfilled, then the first boundary value problem* (35) *is uniquely solvable in the space* $\overset{\circ}{W}{}^1_{2,\alpha}(D)$ *for all* $f\in L_2(D), \frac{f_i}{\sqrt{\lambda_i}}\in L_2(D);\ i=1,\ldots,n.$

Furthermore, we established the estimation of solutions of problem (35).

Theorem 17. *If the conditions of the previous theorem are fulfilled, then for the weak solution* $u(x)$ *of the first boundary value problem* (35) *the following estimation is valid*

$$\|u\|_{W^1_{2,\alpha}(D)} \le c\left(\|f\|_{L_2(D)} + \sum_{i=1}^{n} \left\| \frac{f_i}{\sqrt{\lambda_i}} \right\|_{L_2(D)} \right),$$

where the positive constant depends only on n, d, r, μ, ω *and the vector* α .

In section 3.2 üe study a continuity modulus at the boundary point of the solution of the Dirichlet problem for homogeneous equation *Lu=0* from (35) without minor coefficient with a boundary condition $u = \varphi$ on the boundary of the domain *D* with a boundary function φ continuous on ∂D. Below, $\varepsilon_R^y(k)$, where $y \in R^n$, $R > 0$, $k > 0$, means a closed ellipsoid

$$\left\{ x : \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{R^{\alpha_i}} \le (kR)^2 \right\}.$$

We introduce a notion of capacity of the compact K, strictly internal with respect to the fixed ellipsoid Σ .

Let $V_\Sigma(K) = \{u \in \dot{W}^1_{2,\alpha}, \quad u \ge 1$ on K in the sense $W^1_{2,\alpha}\}$,

$$J_\Sigma(u) = \int_\Sigma \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

The number $cap_\Sigma(K) = \inf_{u \in V_\Sigma(K)} J_\Sigma(u)$ is called a capacity of the compact K with respect to Σ, generated by the operator L. In the following statement Σ means the ellipsoid $\varepsilon_1^0(1)$ and it is assumed that the boundary of the domain *D* contains an origin of coordinates.

Theorem 18. *If* $u \in W^1_{2,\alpha}(D)$ *is a weak solution of the Dirichlet problem for the equation* $Lu = 0$ *, whose coefficients satisfy* *condition (36) and the boundary function* φ *vanishes at the intersection of* ∂D *with* Σ*, then we have the estimation*

$$|u(x)| \le C_1(\alpha,\mu,n)\sup_{\partial D}|\varphi|\exp\times$$

$$\times\left[-C_2(\alpha,\mu,n)\sum_{i=1}^{\ln\frac{1}{|x|}/(2+\alpha^+)} e^{i(n-2+|\alpha|/2)}cap_{\sum}\left(\varepsilon^0_{e^{-i}}\setminus D\right)\right].$$

Section 3.3 is devoted to the estimation of the maximum of the modulus of eigen functions of the Dirichlet problem for elliptic equations containing a big parameter on a part of the domain. More exactly, in the bounded Lipschits domain $D\subset R^n$, $n\ge 2$, divided by the hyperplane into two parts $D^{(1)}=D\cap\{x:x_n>0\}$ and $D^{(2)}=D\cap\{x:x_n<0\}$, we consider eigen-functions of the problem

$$-L_\varepsilon u=\lambda\omega_\varepsilon(x)u,\ u\big|_{\partial D}=0, \tag{42}$$

for the operator L_ε of from (29), whose coefficients satisfy condition (30) with the weight

$$\omega_\varepsilon(x)=\begin{cases}1,\ x\in D^{(1)}\\ \varepsilon^{-1},\ x\in D^{(2)},\varepsilon\in(0,1]\end{cases}. \tag{43}$$

Eigen functions are normed by the equality

$$\int_D u^2\omega_\varepsilon dx=1. \tag{44}$$

Below u_m means the eigen function of problem (42) responding to the eigen value λ_m.

Theorem 19. *If conditions* (30) *and* (43), *are fulfilled, then in the assumption* (44) *for the eigen functions of problem* (42) *we have the estimation*

$$\sup_{x\in D}|u_m(x)|\le C\lambda_m^{\frac{n}{4}}.$$

with the constant C*, dependent only on* n*, domain* D *and constant* γ *from* (30).

In chapter IV we consider nonuniformly degenerating second order parabolic equations. The main results of this chapter are in the author's papers [1,4,5,6,32,39].

In section 4.1 we study a class of second order, divergent structure parabolic equations with nonuniform power degeneration. Unique weak solvability of the first boundary value problem for such equations in Sobolev weight spaces is proved..

Below E_n and R_{n+1} are Euclidean spaces of the points $x=(x_1,\dots,x_n)$ and $(x,t)=(x_1,\dots,x_n,t)$ respectively, $\Omega\subset E_n$ is a bounded domain with the boundary $\partial\Omega$, $0\in\Omega$, T_0 and T are positive numbers $Q_T=\Omega\times(-T_0,T)$.Consider in the cylinder Q_T, with bounded base Ω the first boundary value problem

$$Lu=\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)=f,\ f\in L_2(Q_T), \tag{45}$$

$$u\big|_{S_T}=0, \qquad u\big|_{Q_0}=0, \tag{46}$$

under the assumption that $\left\|a_{ij}(x,t)\right\|$ is a symmetric matrix with the elements measurable in Q_T and for $(x,t)\in Q_T$, $\xi\in E_n$ it is fulfilled the condition of uniform ellipticity

$$\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\mu^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2. \tag{47}$$

$\mu\in(0,1]$ is a constant, where $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$, $\quad |x|_\alpha=\sum_{i=1}^{n}|x_i|^{\bar\alpha_i}$,

$\bar\alpha_i=\dfrac{2}{2+\alpha_i}$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\ge 0$, $i=1,\dots,n$.

The solvability of problem (45)–(46) is studied in the space $\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$, that is determined by the completion in the norm

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)} = \left[vrai\max_{t\in[-T_0,T]} \int_{\Omega} u^2 dx + \sum_{i=1}^{n} \int_{Q_T} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 dxdt \right]^{\frac{1}{2}}$$

of the set of infinitely differentiable in the closure Q_T functions equal to zero near S_T. Below $\alpha^+ < 2,$ is determined as in (38). We get the following result.

Theorem 20. *If the coefficients of the operator* L *satisfy condition* (47) *and* $\alpha^+ < 2,$ *then the first boundary value problem (45)-(46) is uniquely weakly solvable in the space* $\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)$ *for all* $f \in L_2(Q_T)$ *and for the solution of this problem the following estimation is valid*

$$\|u\|_{\overset{\circ}{W}{}^{1,0}_{2,\alpha}(Q_T)} \le C_3(\mu,\alpha,\Omega,)\|f\|_{L_2(Q_T)}.$$

Section 4.2 is devoted to Holder continuity of the solution of the homogeneous parabolic equation (45) in the bounded domain $\Omega \subset R^{n+1}$ under the additional assumption

$$0 \le \alpha_i < \frac{2}{n-1}, i = 1,\dots,n \qquad (48)$$

Preliminarily, by Ω^{ρ} we denote the totality of the points $(x',t') \in \Omega,$ for which the cylinder $\{(x,t): |x-x'| < \rho, t'-\rho^2 < t < t'\}$ is contained in Ω. The following a priori estimation of the Holder norm in the vicinity of the degeneration is proved for weak solutions of the indicated equation.

Theorem 21. *The weak solution of homogeneous equation (45), whose coefficients satisfy conditions (47), (48) are Holder continuous in* Ω *and for any* $\rho > 0$ *the following inequality is valid*

$$\|u\|_{C^{\lambda}(\Omega^{\rho})} \le H\|u\|_{C(\Omega)},$$

wherein λ *depends on* $\alpha_1,\dots,\alpha_n, n$ *, while* H *- in addition on* ρ *.*

In section 4.3 the Harnack inequality in the vicinity of the degeneration point is proved for weak non-negative solutions of the homogeneous equation from the previous section.

To formulate the obtained result, we assume

$$Q(\rho)=(-\rho^2R^2,0)\times\varepsilon^0_{\rho R}(1),\ S(\rho)=$$

$$=\left(-\left(\frac{1}{3}+\rho\right)R^2,-\left(\frac{3}{4}-\rho\right)R^2\right)\times\varepsilon^0_{\rho R}(1)$$

$$P(R)=(-R^2,0)\times\varepsilon^0_R(1).$$

Theorem 22. *If u is a non-negative solution of homogeneous equation (45) in the cylinder P(4R), whose coefficients satisfy condition (47) and (48), then we have the Harnack inequality of the form*

$$\sup_{S\left(\frac{1}{3}\right)} u(x,t)\le c\inf_{Q\left(\frac{1}{3}\right)} u(x,t)$$

with the constant c, independent of u and R.

In section 4.4 the results of section 4.3 are generalized to homogeneous non-uniformly degenerating equations of the form (45) under more general requirements on the functions $\lambda_i(x,t)$, $i=1,\dots,n$ from (47). More exactly, it is assumed that the coefficients of equation (45) in the domain under consideration satisfy condition (47), wherein

$$\lambda_i(x,t)=g_i\left(\rho(x)+\sqrt{|t|}\right),\quad \rho(x)=\sum_{i=1}^{n}\omega_i\left(|x_i|\right), \tag{49}$$

where $g_i(z)=\dfrac{\left(\omega_i^{-1}(z)\right)^2}{z^2}$; $i=1,2,\dots,n.$ The functions $\omega_i(t)$ strictly monotonically increase, $\omega_i(0)=0$, $\omega_i^{-1}(t)$, is a function inverse to the function $\omega_i(t)$ and for $i=1,2,\dots,$

$$\omega_i(2t)\le 2\omega_i(t), \tag{50}$$

$$\left(\frac{\omega_i(t)}{t}\right)^{q-1}\int_0^{\omega_i^{-1}(t)}\left(\frac{\omega_i(z)}{z}\right)^q dz\le c_1 t \tag{51}$$

with some constant $q > n$ and positive constant c_1 independent of t. A simple example of the function ω_i is the function $\omega_i(t) = t^{\alpha_i}$ where

$$\alpha_i \geq \frac{-1+\sqrt{1+4q(q-1)}}{2(q-1)}.$$

Assume

$$S(\rho) = \{(x;t) : |x_i| < \rho\omega_i^{-1}(R), i = 1,2,...,n\} \times (-(1/3+\rho)R^2, -(3/4-\rho)R^2),$$

$$Q(\rho) = \Pi_{\rho R} \times (-\rho^2 R^2, 0);$$

where

$$\Pi_R = \{x : |x_i| < \omega_i^{-1}(R), i = 1,2,...,n\},$$

$$P(R) = \Pi_R \times (-R^2, 0).$$

We get the following result

Theorem 23. *If u is a non-negative solution of equation* (45) *in the cylinder P(4R), whose coefficients satisfy conditions* (47)-(48) *and* (50)-(51)*, then the we have the Harnack inequality*

$$\sup_{S\left(\frac{1}{3}\right)} u \leq C \inf_{Q\left(\frac{1}{3}\right)} u$$

with the constant c, independent of u and R.

In section 4.5 the Harnack inequality obtained in section 4.4, is used to prove the Harnack continuity of weak solutions of the homogeneous equation

Conclusions

In the dissertation the following basic results were obtained:
Linear elliptic equations with a partial Mackenhoupt weight was studied. Holder continuity of solutions was proved.

2. p-Laplacian type nonlinear elliptic equations with a partial Mackenhoupt weight were considered. Holder continuity of solutions and Harnack inequality were proved.

3. $p(x)-$Laplacian type elliptic equations degenerating by a small parameter into a part of the domain were studied, Holder continuity of solutions was proved.

4. p-Laplacian type linear and nonlinear elliptic equations degenerating by a small parameter into a part of the domain were considered. Harnack inequality and Holder continuity of solutions of such weightless equations and equations with a Mackenhoupt weight were proved. p-Laplace equation with variable two-phase index p when the interphase is a hyperplane, was considered separately..

5. Linear nonuniformly degenerating elliptic equations were studied. Sovability of the Dirichlet problem for a class of the Dirichlet problem for a class of second order nonuniformly degenerating elliptic equations was proved.

6.The estimation of continuity of a boundary point of the solution of the Dirichlet problem for second order nonuniformly degenerating elliptic equations was given.

7. The estimation of the modulus of the first eigen function uniform by the parameter was found for a second order linear elliptic equation containing a large parameter on a part of the domain.

8.Weak solvability of the first boundary value problem in Sobolev weight spaces was proved for linear nonuniformly degenerating divergent parabolic equations.

9.The Harnack inequality for the solution of nonuniformly degenerating second order divergent parabolic equations was proved.

10.The Holder inequality of solutions of second order nonuniformly degenerating parabolic equations in the divergent form was shown.

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The author deeply honors the memory of this teacher, the late professor, corr.-member of ANAS, prof. I.T.Mamedov and expresses sincere gratitude to prof. Yu.A.Alkhutov for constant attention to the work.

The defense will be held on **14 January 2022** at **14^{00}** at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of Azerbaijan National of Academy Sciences.

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Dissertation is accessible at the Institute of Mathematics and Mechanics of Azerbaijan National of Academy Sciences Library.

Electronic versions of dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of ANAS.

Abstract was sent to the required addresses on **10 December 2021**.

Signed for print: 29.10.2021
Paper format: 60x84 1/16
Volume: 70416
Number of hard copies: 20