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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVESTIGATION OF THE SOLUTION OF DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS

Specialty: 1211.01 – Differential equations

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and development degree.

Elliptic-parabolic equations are encountered in solving many important applied problems. These equations are from the class of degenerate equations that arise in the theory of small bendings of surfaces of revolution, without the moment theory of shells, in the study of wave phenomena during fractal diffusion, in the equations of gas dynamics, mathematical biology, genetics and medicine, etc.

The first fundamental results in this direction were obtained by F.Tricomi, F.I.Frankl, I.N.Vekua, A.V.Bitsadze. Important applications of equations of this type in gas dynamics were discovered, the importance of equations of mixed type in solving problems arising in the theory of infinitesimal bendings of surfaces was noted, and the extremum principle for equations of mixed type was formulated, from which, in particular, the uniqueness of the solution follows.

Subsequently, these results were developed in the works of K.I.Babenko, A.M. Nakhushev, G.Holmgren, S.Gellerstedt, P.Germain and R.Bader, L.Wolfersdorf, M.L.Krasnov, I.A.Kipriyanova, M.S.Salakhitdinov and Z.Kh.Kadyrov, L.S.Parasyuk, Kh.Najafov and others.

An important role is played by problems of mixed type, which can be described using the operation of fractional integro-differentiation. The role of fractional calculus in the theory of equations of mixed type is related to the problem of finding an analogue of the Tricomi problem in multidimensional domains with a spatially oriented surface of parabolic degeneration. This is the problem posed by A.V.Bitsadze, has developed strongly in recent years. The study of non-local boundary and boundary value problems required the study of fractional calculus, which has applications in degenerate differential equations.

The work of M.V.Keldysh served as a start for the further development of this type of problems.

In the case of multidimensional problems of this type, we especially note the works of G.Fikera, O.A.Oleinik, M.I.Vishik, S.G.Mikhlin, A.M.Ilyin and others.

In this paper, linear divergence and nondivergent degenerate equations of the elliptic-parabolic type are considered, and their main part also degenerates. In this connection, we note the works of M. Franciosi, A.Alvino, G.Trombetti in which the strong solvability of the first boundary value problem for elliptic-parabolic equations in nondivergent form with smooth coefficients was proved. In the works of G.Talenti, I.T.Mammadov and his students, V.Iftimie, G. Fiortio, G.C.Wen proved the unique strong solvability of the first boundary value problem for second order elliptic and parabolic equations in nondivergent form and with discontinuous coefficients under a Cordes type condition, also elliptic-parabolic equations of various types. We also mention works by I.Kohn, L.Nirenberg, O.A.Ladyzhenskaya, V.A.Solonnikov, N.N.Uraltseva, E.M.Landis and others.

H.Gajewski and I.V Skrypnik, P. Benilan and P. Wittbold considered systems of equations of elliptic-parabolic type.

In the works of T.S.Gadjiev and E.R.Gasimova considered nondivergent linear elliptic-parabolic equations with discontinuous coefficients.

The dissertation work is devoted to the study of qualitative properties of linear divergent degenerate elliptic-parabolic equations and the study of the strong solvability of linear non-divergence degenerate equations of elliptic-parabolic type. Therefore, the topic of the dissertation work can be considered relevant.

Object of research and subject. The object of this dissertation is linear divergent degenerate equations of elliptic-parabolic type and linear nondivergent degenerate equations of elliptic-parabolic type. In the dissertation, the qualitative properties of linear divergent degenerate equations of elliptic-parabolic type and the unique strong solvability of the first boundary value problem of linear non-divergent degenerate equations of elliptic-parabolic type are investigated.

Goal and tasks of the study. The aim of this dissertation is to study the qualitative properties of solutions of the linear divergent degenerate equations of the elliptic-parabolic type and to study the single-valued powered (almost everywhere) solvability of the first boundary value problem for linear non-divergent degenerate equations of the elliptic-parabolic type.

Research methods. To study the main results, methods of functional analysis, theory of functions and differential equations in partial derivatives are used.

Basic statements to be defended. The following main provisions are put forward for defense:

- Results of the study of questions about obtaining weighted aprior estimates for solutions of degenerate linear divergent equations of elliptic-parabolic type;
- Results of the study of questions about the study of qualitative properties of solutions of degenerate linear divergent equations of elliptic-parabolic type;
- Results of the study of questions on obtaining coercive estimates for solutions of degenerate linear non-divergent equations of elliptic-parabolic type;
- Results of the study of questions on the proof of the strong solvability of the first boundary value problem for degenerate linear nondivergent equations of elliptic-parabolic type.

Scientific novelty of the study. The following results are obtained:

- Weighted aprior estimates for solutions of degenerate linear divergent equations are obtained;
- The qualitative properties of solutions of degenerate linear divergent equations are studied;
- Coercive estimates are obtained for solutions of degenerate linear;
- The strong solvability of solutions of degenerate linear non-divergent equations of the first boundary value problem is proved.

Theoretical and practical significance of the research. The results obtained in the thesis are theoretical. They can be used in the theory of partial differential equations and in problems of physics and mechanics.

Approbation and application. The main results of the dissertation were reported at the Institute of Mathematics and Mechanics of the Azerbaijan National Academy of Sciences at the seminars of the departments "Functional Analysis" (prof. H.I. Aslanov), "Differential Equations" (prof. A.B. Aliyev), as well as at the XXI International Conference -“Problems of Decision Making under Uncertainties” (PDMU, Ukraine, 2013), International Conference - dedicated to the 55th anniversary of the Institute of Mathematices and Mechanics of the Azerbaijan National Academy of Sciences (Baku, 2014), International Conference - Caucasion Mathematics Conference I (CMCI, Tbilisi, 2014) International Conference -“International Workshop on Operator Theory and Applications-2015” (IWOTA, Tbilisi-2015), International Conference -“Mathematical Analysis, Differential Equations and Their Applications-7” (MADEA-7, Baku, 2015), XXVII International Conference - “Problems of Decision Making under Uncertainties” (PDMU, Batumi, 2016), International Conference - “Operators in Morrey-type Spaces and Applications” (OMTSA, Turkey, 2017).

Author's personal contribution. All conclusions and results obtained belong to the author personally.

Author's publications. 13 works have been published on the topic of the dissertation, a list of which is given at the end of the abstract. Two of the articles were published in international summary and indexing systems (Scopus and Web of Science).

Name of the organization where the dissertation work is carried out. The work was carried out at the Institute of Mathematics and Mechanics of the Azerbaijan National Academy of Sciences.

The total volume of the dissertation with a sign, indicating the volume of the structural units of the dissertation separately.

The total volume of the dissertation work is 218433 characters (title page - 588 characters, content 1121 characters, introduction - 56000 characters, the first chapter - 70000 characters, the second chapter - 90000 characters, conclusions – 724 characters). The list of used literature consists of 82 titles.

THE MAIN CONTENT OF THE DISSERTATION

Thesis consists of an introduction, two chapters and a bibliography. We turn to the description of the summary and main results of the dissertation work.

Chapter 1 is devoted to the first boundary value problem for a linear divergent degenerate equation of the elliptic-parabolic type and the study of the qualitative properties of its generalized solution.

Let Ω limited area in $n-$dimensional Euclidean space R^n, with border $\partial\Omega$. We assume that the border $\partial\Omega \subset C^2$. Let Q_T cylinder $\Omega \times (0,T)$, where $T \in (0,\infty)$. In Q_T the following initial-boundary value problem is considered

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) - \psi(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} +$$

$$+c(x,t)u = 0 \quad (1)$$

$$u(x,t) = f(x,t),\ (x,t) \in \partial\Omega \times (0,T) \quad (2)$$

$$u(x,0) = h(x), \quad x \in \Omega \quad (3)$$

Regarding the coefficients and right-hand sides, the following conditions are assumed: $\left\| a_{ij}(x,t) \right\|$ is a real symmetric matrix with measurable in $\xi \in R^n$ elements, and for any $(x,t) \in Q_T$ and $\xi \in R^n$ inequalities are true

$$\gamma\omega(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \gamma^{-1}\omega(x)|\xi|^2 \quad (4)$$

Where γ- constant from the half-interval $(0,1]$, $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$, $i, j=\overline{1,n}$ measurable functions with respect to $(x,t)\in Q_T$. Also

$$c(x,t)\leq 0,\ c(x,t)\in L_{n+1}(Q_T) \tag{5}$$

$$|b_i(x,t)|^2+k\cdot c(x,t)\leq 0, \quad b_i(x,t)\in L_{n+2}(Q_T). \tag{6}$$

k- is some positive constant.

Here $\omega(x)$ weight function from A_p- Mackenhopt class. For completeness, we give a definition. For $1<p<\infty$ will say that the weight $\omega: R^n\to[0,\infty)$ belongs to the class A_p, if $\omega(x)$ is locally integrable and there exists a constant C, which for any ball $B\subset R^n$ done

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{\frac{1}{p-1}}(x)dx\right)^{p-1}\leq C<\infty,$$

where $|B|$ is the Lebesgue measure of the ball B. We say that $\omega: R^n\to[0,\infty)$belongs to the class A_p if there is a constant C such that $\frac{1}{|B|}\int_B \omega(x)dx\leq C\omega(x)$, for all $x\in B$.

But $\psi(x,t)$ in the following form, the weight function in

$$\psi(x,t)=\omega(x)\cdot\lambda(\rho)\cdot\varphi(T-t), \tag{7}$$

where $\omega(x)\in A_p$ the Mackenhopt class, where $\lambda(\rho)\geq 0$, $\lambda(\rho)\in C^1[0, diam\Omega]$,

$$|\lambda'(\rho)|\leq p\sqrt{\lambda(\rho)}, \text{where } \rho=dist(x,\partial\Omega), \tag{8}$$

$$\varphi(z)\geq 0,\ \varphi'(z)\geq 0,\ \varphi(z)\in C^1[0,T],$$

$$\varphi(z)\geq\beta\cdot z\cdot\varphi'(z), \varphi(0)=\varphi'(0)=0, \tag{9}$$

where $p,\beta-$ are positive constants.

At some points, we require that the weight function has a derivative. It is related to applied tasks. Namely, in the theory of semiconductors, the weight function is chosen so that it has a derivative, where the weight is chosen as the Fermi integral $\sigma = F_{\gamma+1}$, where

$$F_{\gamma}(u) = \frac{1}{\Gamma(\gamma+1)}\int_0^{\infty} \frac{s^{\gamma}\,ds}{1+\exp(s-u)}\,, \gamma > -1\,,$$

and $\omega(u) = \sigma'$.

Such weights also appear in phase separation problems. Namely

$$\sigma(u) = \frac{1}{1+\exp(-u)}\,,\quad \omega(u) = \sigma'(u) = \frac{1}{\left(1+e^{u}\right)\left(1+e^{-u}\right)}.$$

Regarding the right parts (1)-(3), it is assumed that the conditions

$$f(x,t) \in L^{\infty}(Q_T) \cap L_{\infty}\left(0,T;W_2^1(\Omega)\right) \cap L_1\left(0,T;W_{\infty}^1(\Omega)\right)$$

$$\frac{\partial f}{\partial t} \in L_1\left(0,T;L^{\infty}(\Omega)\right) \tag{10}$$

$$h(x) \in L_1\,(\Omega)\,. \tag{11}$$

Without loss of generality, for ease of calculation, we further take $h(x)$ equal to zero.

Let us introduce some Banach spaces of functions defined on Q_T, with finite norms

$$\|u\|_{W_{2,\omega}^1(Q_T)} = \left(\int_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^{n} u_{x_i}^2\right)dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_2^2(Q_T)} = \left(\int_{Q_T} \left(u^2 + \sum_{i=1}^{n} u_{x_i}^2 + \sum_{i,j=1}^{n} u_{x_i x_j}^2\right)dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_2^{2,1}(Q_T)} = \|u\|_{W_2^2(Q_T)} + \|u_t\|_{L_2(Q_T)},$$

$$\|u\|_{W_{2,\varphi}^{2,2}(Q_T)}=\left(\left(\int_{Q_T}\omega(x)\left(u^2+\sum_{i=1}^n u_{x_i}^2+\sum_{i,j=1}^n u_{x_ix_j}^2+u_t^2\right)+\right.\right.$$

$$\left.\left.+\psi^2(x,t)u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{x_it}^2\right)dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_{2,\psi}^{1,2}(Q_T)}=\left(\left(\int_{Q_T}\omega(x)\left(u^2+\sum_{i=1}^n u_{x_i}^2+\sum_{i,j=1}^n u_t^2\right)+\psi^2(x,t)u_{tt}^2\right)dxdt\right)^{\frac{1}{2}}.$$

$\overset{\circ}{W}{}_{2,\psi}^{1,2}(Q_T)$-subspace of space $W_{2,\psi}^{1,2}(Q_T)$ where the dense set is the collection of all functions from $C^\infty(\overline{Q}_T)$ vanishing at the boundary $\Gamma(Q_T)$.

For $R>0$, $x^0\in R^n$ across $B_R(x^0)$ denote the ball $\{x:|x-x^0|<R\}$, and through $Q_T^R(x^0)$- cylinder $B_R(x^0)\times(0,T)$. Let $\overline{B_R}(x^0)\subset\Omega$. Let's say that $u(x,t)\in A(Q_T^R(x^0))$, if $u(x,t)\in C^\infty(\overline{Q}_T^R(x^0))$, $u|_{t=0}=0$ and $\operatorname{supp} u\subset\overline{Q}_T^\rho(x^0)$ for some $\rho\in(0,R)$. Then $u(x,t)\in A_1(Q_T^R(x^0))$ (or for short, $A_1(Q)$), if $u(x,t)\in C^\infty(\overline{Q}_T^R(x^0))$, $u|_{t=0}=0$. And finally $u(x,t)\in B(Q_T^R(x^0))$, if $u(x,t)\in A(Q_T^R(x^0))$ and besides $u|_{t=T}=u_t|_{t=T}=0$.

Next, we denote for $\rho>0$ a bunch of $\{x:x\in\Omega,\ dist(x,\partial\Omega)>\rho\}$ across Ω_ρ, let $Q_T(\rho)=\Omega_\rho\times(0,T)$. Everywhere further the record $C(.)$ means that a positive constant C depends only on the contents of the brackets.

Let function $u(x,t)\in L_2\left(0,T;W_{2,\psi}^{1,2}(\Omega)\right)$ be called generalized solution of problem (1)-(3) if the integral identity

$$\int_{Q_T} \frac{\partial u}{\partial t}\bar{\varphi} dxdt +$$

$$+\int_{Q_T}\left[\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial \bar{\varphi}}{\partial x_i}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}\bar{\varphi}+c(x,t)u\bar{\varphi}\right]dxdt -$$

$$-\int_0^T\int_\Omega \psi(x,t)\frac{\partial^2 u}{\partial t^2}\bar{\varphi} dxdt = 0 \tag{12}$$

for an arbitrary function $\bar{\varphi} \in C^\infty\left(\overline{Q}_T\right)$ vanishing at $\Gamma(Q_T)$, $\bar{\varphi}(\tau, x) = 0$, $x \in \Omega$ for almost every $\tau \in (0,T)$ and almost everywhere for $t \in (0,T)$.

$$u(x,t) - f(x,t) \in L_2\left(0, T; W^{1,2}_{2,\psi}(\Omega)\right). \tag{13}$$

First, an auxiliary problem is considered, which is obtained from problem (1)-(3) by changing the weights $\omega(x)$ and $\psi(x,t)$ regularized $\omega_\varepsilon(x), \psi_\varepsilon(x,t)$. Instead of $\omega(x) = \omega_\varepsilon(x)$, which

$$\omega_\varepsilon(x) = \max\left\{\omega(x), \omega\left(-\frac{x}{\varepsilon}\right)\right\} \tag{14}$$

for $\varepsilon \in (0,T]$, $\omega_0(x) = \omega(x)$.

Instead of $\psi(x,t) = \psi_\varepsilon(x,t)$,which for any fixed $\varepsilon \in (0,T)$ entered like this

$$\psi_\varepsilon(x,t) = \psi(x,\varepsilon) - \frac{\psi'(x,\varepsilon)\varepsilon}{m} + \frac{\psi'(x,\varepsilon)}{m\varepsilon^{m-1}} t^m \; , \; t \in [0,\varepsilon] \tag{15}$$

and $\psi_\varepsilon(x,t) = \psi(x,t)$ at $t \in [\varepsilon, T]$, $m = \dfrac{2}{\beta}$ for almost every $x \in \Omega$.

It's obvious that $\psi_\varepsilon(x,t) \in C^1[0,T]$. At $t \in [0,T]$

$$\psi_\varepsilon(x,t) \geq \frac{1}{2}\psi(x,t)$$

for almost every $x \in \Omega$. It suffices to prove this inequality for $t \in [0, \varepsilon)$.

It is clear that, due to the monotonicity of $\psi(x,t)$, it will be satisfied if

$$\psi(x,\varepsilon) - \frac{\psi'(x,\varepsilon)\varepsilon}{m} \geq \frac{1}{2}\psi_\varepsilon(x,\varepsilon)$$

or

$$\psi(x,\varepsilon) \geq \frac{2}{m}\psi'(x,\varepsilon)\cdot\varepsilon$$

But the last estimate is true due to conditions (9). Everywhere below, we restrict ourselves to the most interesting case, when $\psi(x,t) > 0$ at $x > 0,\ t > 0$. If $\psi(x,t) \equiv 0$, then equation (1) is parabolic and the corresponding results can be obtained from the results for parabolic equations. But if Если $\psi(x,t) \equiv 0$, at $t \in [0, t^0]$, then the solution of problem (1)-(3) can be obtained by gluing the solution $u(x,t)$ in a cylinder Q_{t^0} and solutions $v(x,t)$ initial boundary value problem for a parabolic equation in a cylinder $\Omega \times (t^0, T)$ with boundary conditions $v(x,t^0) = u(x,t^0)$, $v(x,t)\big|_{\partial\Omega\times[t^0,T]} = 0$.

To solve the problem(1)-(3) the following theorem 1 is true.

Theorem 1. Let conditions (4)-(11) be satisfied. Then there is a constant M_1, depending only on known parameters and independent of $\varepsilon \in (0,1]$, such that the solution $u(x,t)$ problem (1)-(3) with weights $\omega_\varepsilon(x)$, $\psi_\varepsilon(x,t)$ satisfies the assessment

$$\underset{t\in(0,T)}{ess}\sup\int_\Omega \{\Lambda_1(u(x,t)) + \Lambda_2(u(x,t))\}dx + \int_{Q_T}\omega_\varepsilon(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt +$$

$$+\int_{Q_T}\psi_\varepsilon(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt \leq M_1 \qquad (16)$$

where

$$\Lambda_1(u)=\int_0^u s\cdot\omega(s)ds\,,\ \Lambda_2(u)=\int_0^u s\cdot\psi(x,s)ds$$

Theorem 2 is also true.

Theorem 2. Let the conditions of theorem 1 be satisfied. Then there exists a constant M_2, depending only on known parameters and independent of $\varepsilon\in(0,1]$, such that the solution of the regularized problem (1)-(3) satisfies the estimate

$$\int_{Q_T}\left[\omega_\varepsilon(x)\left|\frac{\partial u}{\partial x}\right|^2+\psi_\varepsilon(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le M_2\,. \tag{17}$$

To prove theorem 2, we need an auxiliary estimate.

Lemma 1. Assume that the conditions of theorem 1 are satisfied and the following estimate holds

$$\underset{t\in(0,\tau)}{ess\sup}\int_\Omega u^q(x,t)dx\le C_1 \tag{18}$$

for all numbers $q\in\left(\frac{2n}{n+2},\frac{n}{2}\right)$, and C_1- constant which depends only on known parameters. Then the estimate

$$\underset{t\in(0,\tau)}{ess\sup}\left\{\int_\Omega|u(x,t)|^{\frac{pn}{n-2}}dx+\int_\Omega|u(x,t)|^{p-2}\left|\frac{\partial u}{\partial x}\right|dx\right\}\le C_2 \tag{19}$$

for all numbers $p>2$, defined by equality

$$p\cdot\frac{n}{n-2}=(p-1)\frac{q}{q-1} \tag{20}$$

and constant C_2 depends only on known parameters and here $n\ne 2$. For the case $n=2$, the calculations need to be changed.

To prove the regularity properties of the function $u(x,t)$ we also need the following growth condition

$$\rho_1^{-1}\left(u^\gamma+1\right)\le u\le\rho_1\left(u^\gamma+1\right),\ u>0\,,\ 0\le\gamma\le\frac{2}{n-2}\,, \tag{21}$$

with some positive constant ρ_1. From (21) it follows that $u \le \rho_1\left(\frac{u^{\gamma+1}}{\gamma+1}+u\right)$, for $u>0$ with $\gamma+1<\frac{n}{n-2}$. Note that this type of condition arises for $n>2$ along with the restriction $\gamma+1<\frac{2}{n-2}$.

For what follows, we need the following lemmas.

Lemma 2. Assume that the conditions of theorems 1 and 2 are satisfied and the inequality

$$\operatorname{ess}\sup_{t\in(0,\tau)}\int_\Omega u_+^q(x,t)dx+$$

$$+\iint_{(u>1)}\left[\omega_\varepsilon^2(x)u^{q-2}\left|\frac{\partial u}{\partial x}\right|^2+\psi_\varepsilon^2(x,t)u^{q-2}(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le C_3 \qquad (22)$$

with some numbers $q\in\left[\frac{2+\gamma}{1+\gamma},\frac{n}{2}\right]$, C_3 is constant depending only on known parameters. Then there is a positive constant C_4, such that the correct estimate

$$\iint_{(u>1)}\left[\omega_\varepsilon^2(x)u^{q-2+\beta}\left|\frac{\partial u}{\partial x}\right|^2+\psi_\varepsilon^2(x,t)u^{q-2+\beta}(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le C_4 \qquad (23)$$

In proving this lemma, we use theorem 2. and lemma 1.

Lemma 3. Suppose that the conditions of Theorem 2 are satisfied. Then there is a number $\bar{q}$ such that $\bar{q}>\frac{n}{2}$ and constant C_5 depending only on known parameters, that the estimate is true

$$\operatorname{ess}\sup_{t\in(0,\tau)}\int_\Omega u_+^{\bar{q}}(x,t)dx+$$

$$\iint_{\{u>1\}}\left[\omega_\varepsilon^2(x)u^{\bar{q}-2}\left|\frac{\partial u}{\partial x}\right|^2+\psi_\varepsilon^2(x,t)u^{\bar{q}-2}(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\le C_5\,. \qquad (24)$$

Notice, that across $u_+(x,t)$ denote $\max\{u(x,t);0\}$. Finally, in section 1.3 we prove the theorem.

Theorem 3. Let the assumptions of theorem 2 be satisfied. Then for an arbitrary $t \in [0,\tau]$, $x', x'' \in \Omega$ correct estimates

$$\|u(x,t)\|_{L^\infty(Q_T)} \le M_3,$$

$$|u(x',t)\omega(x') - u(x'',t)\omega(x'')| \le M_4 |x' - x''|^\eta \qquad (25)$$

with $\eta \in (0,1)$ and constants M_3, M_4 depending only on known parameters and independent of ε.

Theorem 4. Let conditions (4)-(11), (21) be satisfied and the condition $\omega'(x) \le \rho_2 \omega(x)$, $\rho_2 > 0$ is constant. Then there is a constant M_5 depending only on known parameters and independent of ε that an arbitrary solution to problem (1)-(3) satisfies the estimate

$$ess\sup\{|u(x,t)| : (x,t) \in Q_T\} \le M_5 \qquad (26)$$

Theorem 5. Let the conditions of theorem 4. be satisfied. Then the initial boundary value problem (1)-(3) has at least one solution in the sense of the integral identity (12).

Theorem 6. Let the conditions of theorem 4. be satisfied and, additionally, that the functions $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$ locally Lipschitz with respect to x. Then the initial boundary value problem (1)-(3) has a unique solution.

In this way, the existence of a unique solution is proved.

The proof is carried out in four stages, for the final result we use the Gronwall lemma.

Chapter 2 is devoted to obtaining a priori estimates and strong solvability of the first boundary value problem for linear nondivergent degenerate equations of the elliptic-parabolic type.

The first boundary value problem is considered

$$Zu = \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i x_j} + \psi(x,t) u_{tt} - u_t = f(x,t) \qquad (27)$$

$$u\big|_{\Gamma(Q_T)}=0 \tag{28}$$

Here $\Gamma(Q_T)=(\partial\Omega\times[0,T])\cup(\Omega\times\{(x,t):t=0\})$ - is parabolic boundary of the cylinder $Q_T=\Omega\times(0,T)$, where $T\in(0,\infty)$, and $\psi(x,t)$ tends to zero. Let the coefficients satisfy the following conditions: $\|a_{ij}(x,t)\|$ symmetric matrix and for any $(x,t)\in Q_T$ and $\xi\in R^n$ right

$$\gamma\omega(x)|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\gamma^{-1}\omega(x)|\xi|^2 \tag{29}$$

where $\gamma\in(0,1]$, $a_{ij}(x,t)$, $i,j=\overline{1,n}$ measurable functions $(x,t)\in Q_T$ and $\omega(x)\in A_p$ satisfies the Mackenhoupt condition, and

$$\psi(x,t)=\omega(x)\cdot\lambda(\rho)\cdot\varphi(T-t), \tag{30}$$

where $\rho=\rho(x)=dist(x,\partial\Omega)$, and relatively $\psi(x,t)$ conditions are met:

$$\lambda(\rho)\ge 0,\ \lambda(\rho)\in C^1[0,diam\Omega] \text{ and } |\lambda'(\rho)|\le p\sqrt{\lambda(\rho)}, \tag{31}$$

$$\varphi(z)\ge 0,\ \varphi'(z)\ge 0,\ \varphi(z)\in C^1[0,T],$$

$$\varphi(z)\ge\beta\cdot z\cdot\varphi'(z),\varphi(0)=\varphi'(0)=0, \tag{32}$$

here p and β are positive constants.

Section 2.1 considers the model operator

$$Z_0=\omega(x)\cdot\Delta+\psi(x,t)\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}$$

where $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator .

Lemma 4. If relatively $\omega(x)$ the Mackenhoupt condition is satisfied, and $\psi(x,t)$ condition (30)-(32), then there exists $T_1(\psi(x,t),n,\omega(x))$,such that at $T\le T_1$, for any function $u(x,t)\in B(Q_T^R(x_0))$ fair assessment

$$\int\limits_{Q_T^R(x_0)} \left(\omega(x) \left(\sum_{i,j=1}^{n} u_{x_i x_j} + u_t^2 \right) + \psi_\varepsilon^2(x,t) u_{tt}^2 + \psi_\varepsilon(x,t) \sum_{i=1}^{n} u_{x_i t}^2 \right) dxdt \le$$

$$\le \left(1 + 2(n+1)q(T)\right) \int\limits_{Q_T^R(x_0)} (Z_\varepsilon u)^2 \, dxdt \cdot \qquad (33)$$

Lemma 5. Let with respect to the function $\psi(x,t)$ conditions (30)-(32) are satisfied, $\omega(x)$ satisfies the Mackenhoupt condition, and the operator Z_ε at $\varepsilon > 0$ has the same meaning as in lemma 1. Then for $T_1(\psi(x,t), n, \omega(x))$, for any function $u(x,t) \in \overset{\circ}{W}{}_{2,\psi_\varepsilon}^{2,2}(Q_T)$ fair assessment

$$\|u(x,t)\|_{\overset{\circ}{W}{}_{2,\psi_\varepsilon}^{2,2}(Q_T)} \le C_6(\psi, \omega, n, \Omega) \|Z_\varepsilon u - \mu u\|_{L_2(Q_T)}, \qquad (34)$$

here $\mu = \frac{1}{T}$, $\overset{\circ}{W}{}_{2,\psi_\varepsilon}^{2,2}(Q_T)$ – Banach space of functions $u(x,t)$ given on Q_T with a final rate. And zero from above is the completion of the set of all functions from $C^\infty\left(\overline{Q}_T\right)$vanishing at ∂Q_T, according to the space norm $W_{2,\psi_\varepsilon}^{2,2}(Q_T)$. Some auxiliary lemmas are proved for the solvability of problems for a model operator Z_0.

Lemma 6. Let with respect to the function $\psi(x,t)$ conditions (30)-(32) are satisfied, $\omega(x)$ satisfies the Mackenhoupt condition, then there exists $T_1(\psi, \omega, n)$ such that at $T \le T_1$ for any function $u(x,t) \in A\left(Q_T^R\left(x^0\right)\right)$ fair assessment

$$\int\limits_{Q_T^R\left(x^0\right)} \left(\omega(x) \left(\sum_{i,j=1}^{n} u_{x_i x_j} + u_t^2 \right) + \psi_\varepsilon^2(x,t) u_{tt}^2 + \psi_\varepsilon(x,t) \sum_{i=1}^{n} u_{x_i t}^2 \right) dxdt \le$$

$$\le \left(1 + D \cdot S\right) \int\limits_{Q_T^R\left(x^0\right)} (Z_0 u)^2 \, dxdt. \qquad (35)$$

where $S = S(\psi, \omega, n)$ is some constant $D = D(T) = q(T) + q_1(T)$ $q(t) = \sup\limits_{t\in[0,T]} \varphi'(t)$, $q_1(t) = \sup\limits_{t\in[0,T]} \varphi(t)$.

Lemma 7. If with respect to the coefficients of the operator Z conditions (29) are satisfied, with respect to $\psi(x,t)$ (30)-(32), $\omega(x)$ satisfies the Mackenhoupt condition, then for $T \le T_2$ for any function $u(x,t) \in A\left(Q_T^R\left(x^0\right)\right)$ fair assessment

$$\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi_\varepsilon}\left(Q_T^{R/2}\left(x^0\right)\right)} \le C_7 \left\|Zu\right\|_{L_2\left(Q_T^R\left(x^0\right)\right)} +$$

$$+\varepsilon\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi_\varepsilon}\left(Q_T^R\left(x^0\right)\right)} + \frac{C_8(\psi,\omega,n,\Omega)}{\varepsilon R^2}\left\|u\right\|_{L_2\left(Q_T^R\left(x^0\right)\right)} \qquad (36)$$

for any $\varepsilon > 0$.

Consequence 1. If with respect to the coefficients of the operator Z conditions (29) are satisfied, with respect to $\psi(x,t)$ (30)-(32), $\omega(x)$ satisfies the Mackenhoupt condition, then for $T \le T_2$ and any $\varepsilon > 0$ for every function $u(x,t) \in C^\infty\left(\overline{Q}_T\right), u\big|_{t=0} = 0$ fair score

$$\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T(\rho))} \le C_9(\psi,\omega,\sigma,n,\rho,\Omega)\left\|Zu\right\|_{L_2(Q_T)} +$$

$$+\varepsilon\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)} + C_{10}(\psi,\omega,\sigma,n,\rho,\Omega)\left\|u\right\|_{L_2(Q_T)} \qquad (37)$$

Lemma 8. If with respect to the coefficients of the operator Z conditions (29) are satisfied, with respect to $\psi(x,t)$ (30)-(32), $\omega(x)$ satisfies the Mackenhoupt condition, then there exists $\rho_1(\psi,\omega,n)$ such that if $T \le T_2$, then for any $c > 0$ for every function $u(x,t) \in C^\infty\left(\overline{Q}_T\right)$, $u\big|_{\Gamma(Q_T)} = 0$ fair assessment

$$\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi}\left(Q_T^1(\rho_1)\right)} \le C_{11}(\psi,\omega,\sigma,n,\rho_1,\Omega)\left\|Zu\right\|_{L_2(Q_T)} +$$

$$+\varepsilon\left\|u(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)} + \frac{C_{12}(\psi,\omega,\sigma,n,\rho_1,\Omega)}{\varepsilon}\left\|u\right\|_{L_2(Q_T)} \qquad (38)$$

where $Q_T^1(\rho_1) = Q_T \setminus Q_T(\rho_1)$.

Lemma 9. Under the conditions of lemma 8, for any $u(x,t) \in \overset{\circ}{W}{}_{2,\psi}^{2,2}(Q_T)$ at $T \le T_2$ fair assessment

$$\|u(x,t)\|_{\overset{\circ}{W}{}_{2,\psi}^{2,2}(Q_T)} \le C_{13}(\psi,\omega,n,\Omega)\|Zu\|_{L_2(Q_T)} +$$
$$+ C_{14}(\psi,\omega,n,\Omega)\|u\|_{L_2(Q_T)}. \tag{39}$$

As a result, the coercive estimate is proved.

Theorem 7. If conditions (29), (30)-(32) are satisfied, $\omega(x)$ satisfies the Mackenhoupt condition, then there exists $T_0 = T_0(\psi,\omega,\sigma,n,\Omega)$ such that at $T \le T_0$, for any function $u(x,t) \in \overset{\circ}{W}{}_{2,\psi}^{2,2}(Q)$ correct estimate

$$\|u(x,t)\|_{\overset{\circ}{W}{}_{2,\psi}^{2,2}(Q)} \le C_{15}(\psi,\omega,\sigma,n,\Omega)\|Zu\|_{L_2(Q)}. \tag{40}$$

In the following sections, we consider the issue of strong solvability of the problem posed and prove a unique strong (everywhere) solvability of the first boundary value problem for the operator Z in the corresponding weighted Sobolev space for any right-hand side $f(x,t) \in L_2(Q_T)$. The proof is carried out by the method of continuation with respect to a parameter.

Let us pass to the solvability of the problem for the model operator Z_0, where

$$Z_0 = \omega(x) \cdot \Delta + \psi(x,t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}$$

Lemma 10. If relatively $\omega(x)$ the Mackenhopt condition is satisfied, and $\psi(x,t)$ satisfies conditions (30)-(32), then when $T \le T_5(\psi,\omega)$, $\tau \in [0,1]$ for every function $u(x,t) \in A\left(Q_T^R\left(x^0\right)\right)$ correct estimate

$$\int_{Q_T^R(x^0)} \left(\omega^2(x)\left(\sum_{i,j=1}^n u_{x_i x_j}^2 + u_t^2 \right) + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{x_i t}^2 \right) dxdt \le$$

$$\leq \left(1 + D(T) \cdot S_2\right) \int\limits_{Q_T^R\left(x^0\right)} \left(Z_0 u - \frac{\tau}{T} u \right)^2 dxdt. \qquad (41)$$

where $S_2 = S_2(\psi, \omega, n)$ is some constant $D = D(T) = q(T) + q_1(T)$, $q(t) = \sup\limits_{t \in [0,T]} \varphi'(t)$, $q_1(t) = \sup\limits_{t \in [0,T]} \varphi(t)$.

Lemma 11. Let with respect to the coefficients of the operator Z conditions (29) are satisfied, the weights $\psi(x,t)$ (30)-(32), $\omega(x)$ satisfies the Mackenhopt condition. Then for every function $u(x,t) \in C^{\infty}\left(\overline{Q}_T\right)$, $u\big|_{\Gamma(Q_T)} = 0$ at $T < T_6(\gamma, \sigma, \psi, \omega, n, \Omega)$ and any $\tau \in [0,1]$ fair assessment

$$\left\| u(x,t) \right\|_{W_{2,\psi}^{2,2}(Q_T)} \leq C_{16}(\gamma, \sigma, \psi, \omega, n, \Omega) \left\| Zu - \frac{\tau}{T} u \right\|_{L_2(Q_T)} \qquad (42)$$

Let us show the strong solvability of the first boundary value problem. First, we give a theorem on the strong solvability of an auxiliary operator. A cross M_0 denote the operator $Z_0 - \mu$, and through T_0 – the minimum value is constant in lemma 10 and lemma 11.

Theorem 8. Let relatively $\psi(x,t)$ conditions (30)-(32) are satisfied, $\omega(x)$ satisfies the Mackenhoupt condition. Then at $T \leq T^0$ first boundary value problem

$$M_0 u = f(x,t), \ (x,t) \in Q_T$$

$$u\big|_{\Gamma(Q_T)} = 0$$

uniquely strongly resolvable in space $\overset{\circ}{W}{}_{2,\psi}^{2,2}(Q_T)$ for every function

$$f(x,t) \in L_2(Q_T).$$

We now present a theorem on the strong solvability of the main problem.

Theorem 9. Let with respect to the coefficients of the operator Z conditions (29) are satisfied, the weights $\psi(x,t)$ (30)-(32) and $\omega(x)$ the Mackenhopt condition. Then at $T \le T^0$ the first boundary value problem (27)-(28) is uniquely strongly solvable in the space $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ for every function $f(x,t) \in L_2(Q_T)$. And for solving $u(x,t)$ fair assessment

$$\|u(x,t)\|_{W^{2,2}_{2,\psi}(Q_T)} \le C_{17} \|f\|_{L_2(Q_T)}. \tag{43}$$

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CONCLUSIONS

The dissertation work is devoted to the study of qualitative properties of linear divergent degenerate elliptic-parabolic equations and the study of the powered solvability of linear nondivergent degenerate equations of elliptic-parabolic type.

The following results are obtained:

- Weighted aprior estimates for solutions of degenerate linear divergent equations are obtained;
- The qualitative properties of solutions of degenerate linear divergent equations are studied;
- Coercive estimates are obtained for solutions of degenerate linear nondivergent equations;
- The strong solvability of solutions of degenerate linear non-divergent equations of the first boundary value problem is proved.

The basic results of the dissertation work are in the following works:

1. Gadjiev, T., Kerimova, M. The solutions degenerate elliptic-parabolic equations// -India: Journal of Advances in Mathematics, -2013. v.3, №3, -pp. 219-235.
   Gadjiev, T., Kerimova, M. On some estimations of solutions for degenerate elliptic-parabolic equations// -Baku: Transactions of ANAS, issue mathematics mechanics, series of phys.-tech. & math. sc. - 2013. XXXIII, №4, -pp. 57-72.
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