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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

RIEMANN PROBLEMS IN HARDY CLASSES WITH VARIABLE SUMMABILITY INDEX AND THEIR APPLICATIONS IN BASICITY PROBLEMS

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GENERAL CHARACTERISTICS OF WORK

Relevance and degree of development of the topic. Almost all branches of natural science face the problems of approximating large scale objects by the lesser or simpler ones. Such problems arise, for example, in theory of equations (e.g., Fourier method for solving partial differential equations), in spectral theory of linear operators, in harmonic analysis, in theory of nonharmonic Fourier series (nonharmonic analysis), in signal theory, in frame theory, in wavelet analysis, in image recognition theory, etc. Approximation by perturbed trigonometric systems in different function spaces is one of such problems. For example, when solving many partial differential equations of elliptic or mixed type in special domains there arise perturbed systems of sines or cosines, which, in turn, requires the study of basis properties (completeness, minimality, basicity, etc.) of these systems in various function spaces induced by differential equations. Note that this field has a long and rich history. Classical theorems of Runge and Walsh (as well as Weierstrass theorem) on approximation of continuous functions on the (closed or open) Jordan curves by polynomials or rational functions can be attributed to this field.

Since recently (from the late 20th Century), in the context of some problems of mathematics (for example, in the study of smoothness properties of the Jacobian of mapping), mechanics and mathematical physics, there has been great interest in the study of different problems in non-standard function spaces, such as Lebesgue spaces with variable summability index, Morrey-type spaces, grand spaces, etc. A lot of monographs and review articles have been dedicated to this field. So there arises a necessity to study approximation problems in these spaces.

This thesis is dedicated to the study of basis properties of the single exponential system

$$a(t)e^{\mathrm{int}} - b(t)e^{-\mathrm{int}}, \qquad n \in N, \tag{1}$$

with complex-valued coefficients $a(t)=|a(t)|e^{i\alpha(t)}$, $b(t)=|b(t)|e^{i\beta(t)}$, the perturbed exponential system

$$\exp[i(nt+\gamma(t)signn)],\ n\in Z, \tag{2}$$

and the system of cosines

$$1\cup\cos(nt+\gamma(t)),\ \ n\in N, \tag{3}$$

where $\gamma(\cdot)$ is in general a complex-valued function, in the Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ (in cases (1) and (3)) and $L_{p(\cdot)}(-\pi,\pi)$ (in case (2)) with variable summability index $p(\cdot)$, respectively. Trigonometric systems are special cases of the systems (1)-(3). The interest in various special cases of the sequence (1) is aroused by their significant use in many applied problems. For example, when solving the problems of spectral theory of differential operators, A.V.Bitsadze, V.A.Ditkin, S.M.Ponomarev and E.I.Moiseev had to consider the approximation properties of very special cases of the system (1).

Special cases of the system $\{\sin(nt-\alpha(t))\}_1^{\infty}$, where $\alpha(t)$ is a piecewise Hölder function on the interval $[0,\pi]$, occur in the study of the equations of hydrodynamics. Gabov S.A. and Krutitski P.A. considered a non-stationary problem of diffraction of stable internal wave on a barrier installed on the bottom of a channel, when the wave exitation is happening due to small fluctuations of the barrier. So, in Cartesian coordinates $(x;z)$, the dynamics of small two-dimensional motions of a fluid exponentially stratified along the z axis in Boussinesq approximation is described by the Sobolev equation

$$\frac{\partial^2}{\partial t^2}\Delta u+\omega_0^2 u_{xx}=0;\ \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

The considered fluid fills the channel

$$\Omega=\{(x;z):-\infty<x<+\infty,\ 0<z<\pi\}.$$

Let

$$I=[0,\pi],\ I_1=[0,\alpha],\ \Gamma=\{(x;z):\ x=0,\ z\in I\},$$
$$\Gamma_1=\{(x;z):\ x=0,\ z\in I_1\},$$

Problem. Find the function $u(x;z;t)$, whish is defined and continuous on $\overline{\Omega}\times[0,\infty)$, satisfies the equation $\frac{\partial^2}{\partial t^2}\Delta u+u_{xx}=0$ in $(\Omega\setminus\Gamma)\times(0,\infty)$ in the classical sense, initial conditions $u(x;z;0)=u_t(x;z;0)\equiv 0$, $\forall(x;z)\in\Omega$, boundary conditions $u/_{z=0}=u/_{z=\pi}=0$ on the walls of the channel Ω and boundary condition $u=f(t;z)$ on Γ_1, where $f(t;z)\in C_0^{(2)}\ [[0,\infty),C^{(2,\lambda)}(I_1)]$ and $\lambda\in(0,1]$ is a Hölder coefficient.

To construct the explicit solution of this problem, the expansion of the given function with respect to the system

$$V_n(z)=\begin{cases}\sin nz,\ z\in I_1;\\ \cos nz,\ z\in I\setminus I_1\end{cases}$$

is used. The study of similar problem in Sobolev spaces requires the knowledge of basicity properties of the system $\{V_n\}_{n\in N}$ in the corresponding function spaces.

Basis properties of the systems (1)-(3) in the Lebesgue spaces L_p, $1<p<+\infty$, have been fully studied by B.T.Bilalov. Basicity of the systems of sines $\{\sin(n-\alpha)t\}_{n\in N}$ and the systems of cosines $\{\cos(n-\alpha)t\}_{n\in Z_+}$, where $\alpha\in R$ is a real parameter, in the Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ with variable summability index $p(\cdot)$ has been studied by B.T.Bilalov and Z.G.Huseynov. These authors first consider the basicity of the exponential system $\{\exp i(n-\alpha\, signn)t\}_{n\in Z}$ in $L_{p(\cdot)}(-\pi,\pi)$, and then, using the obtained results, they find the basicity conditions of corresponding systems of sines and cosines. T.R.Muradov considered the exponential system $\{\exp i(nt-\alpha(t) sign\, n)\}_{n\in Z}$, where $\alpha(\cdot)$ is a piecewise Hölder function on $[-\pi,\pi]$, and found a sufficient condition for the basicity of this system in the spaces $L_{p(\cdot)}(-\pi,\pi)$. N.P.Nasibova considered the exponential system $\{\exp i(nt+\gamma(t)signn)t\}_{n\in Z}$, where $\gamma(t)=\alpha t+\beta signt$, $\alpha;\beta\in R$, is a piecewise linear function. She found sufficient conditions on the parameters α and β, which provide that this system forms a basis for the weighted space $L_{p(\cdot),\rho}(-\pi,\pi)$ when the weight $\rho(\cdot)$ has an exponential form.

In this thesis, the systems (1)-(3) are considered in the Lebesgue spaces $L_{p(\cdot)}$ with variable summability index $p(\cdot)$. We consider the case where $\arg a(t), \arg b(t)$ and $\gamma(t)$ are piecewise continuous functions which may have an infinite number of discontinuity points of the first kind. To study the basis properties of these sysems, we use Runge's method of boundary value problems of the theory of analytic functions. Unlike ordinary cases, our boundary value problem has some particularities. Namely, the coefficient and the right-hand side of the problem satisfy some conditions. This, in turn, affects the solvability of this problem in the Hardy classes with variable summability index and the basicity conditions of the systems (1)-(3) in the spaces $L_{p(\cdot)}$.

Object and subject of research Lebesgue spaces with variable summability index, Hardy classes with variable summability index, Riemann boundary value problems, basicity of perturbed trigonometric systems.

The goal and objectives of the study. The main purpose of this research is to establish the solvability of the Riemann boundary value problems with Carleman shift in the Hardy classes with variable summability index and to find necessary and sufficient conditions on the jumps of arguments of the coefficients of the problem which provide the basicity of the considered single exponential systems in the Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ with variable summability index $p(\cdot)$.

General technique of studies. Our approach to the considered problem dates back to A.V.Bitsadze. It consists of reducing the original problem to some conjugation problem in corresponding classes of analytic functions. This approach has been successfully used by S.M.Ponomarev, A.N.Barmenkov, G.G.Devdariani, B.T.Bilalov, etc. Research method used in this work is a generalization of the method developed by B.T.Bilalov.

Main provisions of dissertation. Chapter 1 of this thesis is a preparation for obtaining main results. It contains definitions of Lebesgue spaces and Hardy classes with variable summability index and their basic properties. In this chapter we consider a special Riemann problem with Carleman shift on the unit circumference and we find sufficient conditions on the coefficients of this problem which provide the solvability of this problem in the considered Hardy classes.

In Chapter 2, the results obtained in Chapter 1 are applied to the basicity problems of single exponential systems in the Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ with variable summability index $p(\cdot)$ on $(0,\pi)$. These systems are the generalizations of perturbed systems of sines and cosines arising in the theory of differential equations.

In Appendix, an analog of the well-known Riesz theorem for the classes H_p^+ is considered. Namely, its validity for an arbitrary measurable subset of $(-\pi,\pi)$ is proved.

Scientific novelty. The following main results are obtained in this work:

- homogeneous Riemann problem is considered in the case where the coefficient of the problem satisfies some condition on a semicircumference and the general solution of this problem is constructed;

- nonhomogeneous Riemann problem is considered in the case where the coefficient of the problem and its right-hand side satisfy some conditions on a semicircumference and the sufficient condition is found which provides the solvability of this problem in the Hardy classes with variable summability index;

- single exponential system (1) with the complex-valued coefficients $a(t)$, $b(t)$ is considered in the case where the arguments

$\arg a(t)$ and $\arg b(t)$ may have an infinite number of discontinuity points of the first kind on $[0,\pi]$ and the sufficient and necessary condition on the jumps of the function $\arg a(t) - \arg b(t)$ is found which provides that this system forms a basis for $L_{p(\cdot)}(0,\pi)$;

- exponential system (2) with a piecewise continuous phase $\gamma(\cdot)$, which is odd on $[-\pi,\pi]$ and may have an infinite number of discontinuity points of the first kind, is considered on $[-\pi,\pi]$ and the sufficient condition on the jumps of the function $\gamma(\cdot)$ is found which provides that this system forms a basis for $L_{p(\cdot)}(-\pi,\pi)$;

- system of cosines (3) with a piecewise continuous phase $\gamma(\cdot)$ is considered and, using the results obtained for the exponential system (2), the sufficient and necessary condition for the basicity of (3) in $L_{p(\cdot)}(0,\pi)$ is found.

Theoretical and practical value of the study. Results obtained in this thesis can be used in the construction of explicit solution of some problems of hydrodynamics, in spectral theory of non-self-adjoint differential operators, in approximation theory, in theory of nonharmonic Fourier series, etc.

Approbation and application. The main results of this work have been presented in the seminar of the Department of "Nonharmonic Analysis" of the Institute of Mathematics and Mechanics of the NAS of Azerbaijan led by professor B.T.Bilalov, in the 7-th International Conference on Mathematical Analysis, Differential Equation & Their Applications MADEA-7, 2015, Baku, International Workshop on Non-Harmonic Analysis and Differential Operators, 2016, Baku, in the International Conference on Operators, Functions, and Systems of Mathematical Physics, 2019, Baku, and in the 4-th International Conference on Mathematical Advances and Application, 2021, Istanbul.

Personal contribution of the author. All the results obtained in the work are the personal contribution of the author.

Publications of the author. The main results of the work- 6 of them were published in the recommended journals of the EAC under the President of the Republic of Azerbaijan (3 of them in Scopus and 1 Zentralblatt MATH), conference materials -5 (5 of them were published on international journals, 2 - abroad).

The name of the institution where the dissertation was completed. The work was performed at the department of "Mathematical analysis" of the Ganja State University.

Volume and structure of the dissertation (in signs, indicating the volume of each structural unit separately).

The dissertation consists ~ 201489 characters (title page – 395, content ~ 1094 and introduction ~ 56000 знаков characters, I chapter ~ 60000 character, II chapter ~ 82000 character and conclusion – 4000 character) and the list of used literature consists of 97 items.

THE CONTENT OF THE DISSERTATION

This thesis consists of introduction, two chapters and reference list.

In the introduction, the relevance of the dissertation is substantiated, a brief summary of the results related to the content of the work is given and the main results are commented.

Chapter 1 is fully dedicated to the solvability of special Riemann boundary value problems in the Hardy classes with variable summability index. First, the homogeneous Riemann problem is considered in the case where the coefficient of the problem satisfies some condition on $[-\pi,\pi]$, in other words, the values of the coefficient on $(-\pi,0)$ are defined by its own values on $(0,\pi)$ in a special way. It is assumed that the argument of the coefficient may have an infinite number of discontinuity points of the first kind. Under some conditions on the argument of the coefficient, the general solution of the homogeneous Riemann problem is constructed. Unlike the usual case, the special form of the coefficient allows expanding the condition on the argument of the coefficient. Then the nonhomogeneous Riemann problem is considered in the case where the coefficient and the right-hand side of the problem satisfy special conditions. Sufficient condition for solvability of nonhomogeneous problem in the Hardy classes with variable summability index is found in case where the right-hand side belongs to the Lebesgue space $L_{p(\cdot)}(0,\pi)$. These conditions differ from those in cases where the coefficient and the right-hand side of the problem are defined independently on the whole of $[-\pi,\pi]$.

In **1.1**, standard notation, basic concepts of basis theory and some facts from the theory of Cauchy-type integrals are stated.

We will need some facts from the theory of Lebesgue spaces with variable summability index. Let $p:[-\pi,\pi]\to[1,+\infty)$ be some Lebesgue-measurable function. Denote the class of all measurable (with respect to Lebesgue measure) functions on $[-\pi,\pi]$ by $\mathcal{L}_0$. Also denote

$$I_p(f)\overset{def}{\equiv}\int_{-\pi}^{\pi}|f(t)|^{p(t)}dt.$$

Let

$$\mathcal{L}\equiv\{f\in\mathcal{L}_0 : I_p(f)<+\infty\}.$$

With regard to ordinary linear operations of adding functions and multiplication by a number, for $p^+=\sup\underset{[-\pi,\pi]}{vrai}\,p(t)<+\infty$, $\mathcal{L}$ becomes a linear space. With respect to the norm

$$\|f\|_{p(\cdot)}\overset{def}{\equiv}\inf\left\{\lambda>0 : I_p\left(\frac{f}{\lambda}\right)\le 1\right\},$$

$\mathcal{L}$ is a Banach space, denoted by $L_{p(\cdot)}$. Let

$$WL \overset{def}{\equiv} \left\{ p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \frac{1}{2} \Rightarrow \right.$$
$$\left. \Rightarrow |p(t_1) - p(t_2)| \le \frac{C}{-\ln|t_1 - t_2|} \right\}.$$

Throughout the work, $q(\cdot)$ denotes the function conjugate to $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf \underset{[-\pi,\pi]}{vrai}\, p(t)$, $p^+ = \sup \underset{[-\pi,\pi]}{vrai}\, p(t)$.

Define the weighted Hardy classes $H^{\pm}_{p(\cdot),\rho}$. By $H^+_{p_0}$ we denote the usual Hardy class, where $p_0 \in [1, +\infty)$ is some number. Let

$$H^{\pm}_{p(\cdot),\rho} \equiv \left\{ f \in H^+_1 : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^+ are nontangential boundary values of $f(\cdot)$ on $\partial\omega$.

Similar to the classical case, we define the weighted Hardy class ${}_mH^-_{p(\cdot),\rho}$ of analytic functions in $C \setminus \overline{\omega}$, which have a degree $k \le m$ at infinity. Let $f(z)$ be an analytic function in $C \setminus \overline{\omega}$ with the finite degree $k \le m$ at infinity, i.e.

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k \le m$, and $f_2(z)$ is a principal part of Laurent expansion of $f(z)$ in the neighborhood of an infinitely remote point. If the function $\varphi(z) \equiv \overline{f_2\left(\frac{1}{\overline{z}}\right)}$ belongs to the class $H^+_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_mH^-_{p(\cdot),\rho}$.

Consider the following Riemann problem in the classes $H^+_{p(\cdot),\rho} \times {}_mH^-_{p(\cdot),\rho}$:

$$F^+(\tau) - G(\tau)F^-(\tau) = f(\tau), \tau \in \partial\omega, \tag{4}$$

where $f \in L_{p(\cdot),\rho}$ is some function. By solution of the problem (4), we mean a pair of analytic functions

$(F^+(z);F^-(z)) \in H^+_{p(\cdot),\rho} \times_m H^-_{p(\cdot),\rho}$, boundary values of which satisfy the equality (4) a.e. on $\partial\omega$.

We will also need the concept of conjugation problem with shift. Let $\partial\omega$ be a unit circumference and $\eta : \partial\omega \leftrightarrow \partial\omega$ be some shift. By shift on $\partial\omega$, we mean a reciprocally inverse and continuous mapping of $\partial\omega$ onto itself. By orientation on $\partial\omega$, we mean a positive direction, i.e. direction with respect to which the unit ball stays on the left side. Shifts which change their orientation are usually called Carleman shifts. So, there are two kinds of shifts: Carleman shifts and non-Carleman shifts.

Let $\eta : \partial\omega \to \partial\omega$ be some Carleman shift. Consider the following conjugation problem with shift:

$$F_1^+(\tau) - G(\tau)F_2^+(\eta(\tau)) = g(\tau), \tau \in \partial\omega. \quad (5)$$

By solution of the problem (5) in the Hardy classes $H_p^+ \times H_p^+$, we mean a pair of functions $(F_1;F_2) \in H_p^+ \times H_p^+$, nontangential boundary values of which satisfy the relation (5) a.e. on $\partial\omega$.

In **1.2**, a special homogeneous Riemann boundary value problem is considered, the coefficient of which has an infinite number of discontinuities. Under some conditions on the coefficient of the problem, the Noetherness of this problem is proved and its general solution in the Hardy classes with variable summability index is constructed.

Let the functions $a(\cdot)$ and $b(\cdot)$ satisfy the following conditions:

i) $a^{\pm 1}(\cdot); b^{\pm 1}(\cdot) \in L_\infty(0,\pi)$;

ii) $\alpha(\cdot); \beta(\cdot)$ are piecewise continuous functions on $(0,\pi)$ with discontinuity points $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$, respectively. Assume that the set $\{\tilde{s}_k\} \equiv \{t_k\} \cup \{\tau_k\}$ may have only one limit point $\tilde{s}_0 \in (0,\pi)$ and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has finite left and right limits at the point $\tilde{s}_0$.

iii) $\sum_{k=1}^{\infty}\left|h(\tilde{s}_k)\right| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ are the jumps of the function $\tilde{\theta}(\cdot)$ at the points $\tilde{s}_k$.

iv) the jumps $\{\tilde{h}_i\}$ satisfy the relation $\left(\frac{\tilde{h}(\tilde{s}_i)}{2\pi} + \frac{1}{p(\tilde{s}_i)}\right) \notin Z$, $\forall i \in N$.

Define

$$G(e^{it}) = \begin{cases} b(t)a^{-1}(t), 0 < t < \pi, \\ \\ a(-t)b^{-1}(-t), -\pi < t < 0. \end{cases}$$

Consider the following homogeneous Riemann boundary value problem in the classes $H^{+}_{p(\cdot)} \times_m H^{-}_{p(\cdot)}$:

$$F^{+}(\tau) + G(\tau)F^{-}(\tau) = 0, \quad \tau \in \partial\omega.$$

Denote $\theta(t) = \arg G(e^{it})$. Assume that the following inequalities hold:

$$-\frac{2\pi}{p(s_i)} < \tilde{h}_i < \frac{2\pi}{q(s_i)},\ i = \overline{1,\infty},$$

$$-\frac{\pi}{p(0)} < \alpha(0) - \beta(0) < \frac{\pi}{q(0)}. \tag{6}$$

The lemma below is true.

Lemma 1. *Let* $\{\tilde{s}_k\} \subset [-\pi, \pi]$ *be a set of different points which may have only one limit point* $\tilde{s}_0 \in (-\pi, \pi)$, *and let the set of numbers* $\{\tilde{h}_k\}$ *satisfy the condition iii) and the inequalities (6). Then the infinite product*

$$\psi(s) = \prod_{k=0}^{\infty}\left\{\sin\left|\frac{s - \tilde{s}_k}{2}\right|\right\}^{\frac{h(\tilde{s}_k)}{2\pi}},\ \ s_0 = 0,$$

belongs to the space $L_{p(\cdot)}(-\pi, \pi)$.

Consider the following functions $X_1(\cdot)$ analytic inside (with the «+» sign) and outside (with the «-» sign) the unit circle ω:

$$X_1(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z} dt \right\},$$

$$X_2(z) = \exp\left\{ \frac{i}{4\pi} \int_{-\pi}^{\pi} \theta(t) \frac{e^{it}+z}{e^{it}-z} dt \right\}.$$

Define

$$Z_k(z) \equiv \begin{cases} X_k(z), & |z|<1, \\ \left[X_k(z)\right]^{-1}, |z|>1, \ k=1,2; \end{cases}$$

and let

$$Z(z) = Z_1(z) Z_2(z).$$

The function $Z(\cdot)$ is called a canonical solution of the homogeneous problem

$$F^+(\tau) - G(\tau) F^-(\tau) = 0, \quad \tau \in \partial\omega. \tag{7}$$

The following theorem is proved.

Theorem 1. *Let the conditions i)-iii) hold with respect to the coefficient* $G\left(e^{it}\right)$ *of the problem (7),* $p \in WL, 1 < p^- \le p^+ < +\infty$, *and the jumps* $\{h_k\}_0^\infty$ *of the function* $\arg G\left(e^{it}\right)$ *satisfy the inequalities*

$$\left. \begin{array}{l} -\dfrac{1}{q(s_k)} < \dfrac{h_k}{2\pi} < \dfrac{1}{p(s_k)} \quad , \ k \in N; \\ \\ -\dfrac{1}{q(\pi)} < \dfrac{h_0}{2\pi} < \dfrac{1}{p(\pi)} \quad . \end{array} \right\}$$

Then the general solution of homogeneous Riemann problem (7) in the classes $\left(H^+_{p(\cdot)}; {}_mH^-_{p(\cdot)}\right)$ *has a form* $F(z) \equiv Z(z) P_m(z)$, *where* $Z(z)$ *is a canonical solution of homogeneous problem, and* $P_m(z)$ *is an* *arbitrary polynomial of degree* $\le m$.

In **1.3**, a nonhomogeneous Riemann boundary value problem of the theory of analytic functions, adapted to the system (1), is considered. The case where the argument of the coefficient of the boundary value problem is piecewise continuous on the unit circumference and may have an infinite number of discontinuity points of the first kind is treated. Under some conditions on the corresponding jumps of the argument, the solvability of nonhomogeneous Riemann problem in the Hardy classes with variable summability index is studied.

Consider the following nonhomogeneous Riemann problem:

$$\begin{cases} F^{+}(\tau)+G(\tau)F^{-}(\tau)=\Psi(\tau), \quad \tau\in\gamma, \\ F(\infty)=0, \end{cases} \tag{8}$$

where

$$\Psi(t)=\begin{cases} A^{-1}(t)\psi(t), \ \ t\in(0,\pi), \\ -B^{-1}(-t)\psi(-t), \ t\in(-\pi,0), \end{cases}$$

and $\psi\in L_{p(\cdot)}(0,\pi)$ is some function. Assume that the following inequalities hold:

$$-\frac{1}{q(s_k)}<\frac{h_k}{2\pi}<\frac{1}{p(s_k)}, \forall k;$$

$$-\frac{1}{p(\pi)}<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<\frac{1}{q(\pi)}. \tag{9}$$

The following theorem is proved.

Theorem 2. *Let* $p\in WL\wedge p^{-}>1$ *and the functions* $a(\cdot),b(\cdot)$ *satisfy the conditions i)-iii). If the inequalities (9) hold and the relations*

$$\frac{1}{p(0)}-2<\frac{\beta(0)-\alpha(0)}{\pi}<\frac{1}{p(0)},$$

$$-\frac{1}{p(\pi)} < \frac{\beta(\pi)-\alpha(\pi)}{\pi} < -\frac{1}{p(\pi)} + 2,$$

are true, then the Cauchy-type integral

$$F(z) = \frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{\Psi(\sigma)}{Z^{+}\left(e^{i\sigma}\right)}\frac{d\sigma}{1-e^{i\sigma}z}$$

is a solution of the Riemann problem (8) in the classes $H^{+}_{p(\cdot)} \times_{-1} H^{-}_{p(\cdot)}$.

Chapter 2 is fully dedicated to the study of basis properties of the systems (1)-(3) in the Lebesgue spaces $L_{p(\cdot)}$ with variable summability index $p(\cdot)$. First, a single exponential system (1) with complex-valued coefficients $a(t)$ and $b(t)$ is considered on $[0,\pi]$. It is assumed that the functions $\arg a(t)$ and $\arg b(t)$ may have an infinite number of discontinuity points of the first kind on $(0,\pi)$. The basicity problem of the system (1) in $L_{p(\cdot)}(0,\pi)$ is reduced to the solvability of special Riemann boundary value problem in the Hardy classes with variable summability index. The results obtained in Chapter 1 are applied to the considered boundary value problem and the necessary and sufficient condition on the jumps of the function $\arg a(t) - \arg b(t)$ is obtained which provides that the system (1) forms a basis for $L_{p(\cdot)}(0,\pi)$. These results are then applied to the special cases of the system (1).

In **2.1**, a single exponential system with complex-valued coefficients is studied. Consider the following single exponential system:

$$\nu_n(t) \equiv a(t)e^{\text{int}} - b(t)e^{-\text{int}},\ n \in N,$$

with complex-valued coefficients $a(\cdot); b(\cdot) : [0,\pi] \to C$.

Assume that for some n_0 the inequality

$$\frac{1}{p(0)} + 2(n_0 - 1) < \frac{\beta(0)-\alpha(0)}{\pi} < \frac{1}{p(0)} + 2n_0 \qquad (10)$$

holds. Using the condition iv), define the integers $n_i, i = \overline{1,r},$ from the relations

$$-\frac{1}{p(s_i)}<\frac{h(s_i)}{2\pi}+n_i-n_{i-1}<\frac{1}{q(s_i)},i=\overline{1,r}. \qquad (11)$$

The following main theorem is true.

Theorem 3. *Let the coefficients $a(\cdot)$ and $b(\cdot)$ of the system $\{v_n\}_{n\in N}$ satisfy the conditions i)-iv), and the integers $\{n_i\}_1^r$ be defined by (10), (11). Assume that*

$$\frac{\beta(\pi)-\alpha(\pi)}{2\pi}+\frac{1}{2p(\pi)}\notin Z.$$

If the inequality

$$-\frac{1}{p(\pi)}+2n_r<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<-\frac{1}{p(\pi)}+2(n_r+1)$$

holds, then the system $\{v_n\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(0,\pi)$. Besides, if

$$\beta(\pi)-\alpha(\pi)<-\frac{\pi}{p(\pi)}+2n_r\pi,$$

then the system $\{v_n\}_{n\in N}$ is not complete, but minimal in $L_{p(\cdot)}(0,\pi)$. And, if

$$\beta(\pi)-\alpha(\pi)>-\frac{\pi}{p(\pi)}+2(n_r+1)\pi,$$

then this system is complete, but not minimal in $L_{p(\cdot)}(0,\pi)$.

In **2.2**, an exponential system (2) with a piecewise continuous and odd phase $\gamma(t)$ is considered on $[-\pi,\pi]$. Using the results of Section 2.1, a necessary and sufficient condition for the basicity of this system for the spaces $L_{p(\cdot)}(-\pi,\pi)$ is obtained.

Consider the following exponential system:

$$\varphi_n(\theta)\equiv\exp[i(n\theta-\operatorname{sgn} n\,\alpha(\theta))],\quad n=\pm1,\pm2,..., \qquad (12)$$

where $\alpha(\theta)$ is a piecewise continuous, odd function on $[-\pi,\pi]$, i.e. $\alpha(-\theta)=-\alpha(\theta)$, $\forall\theta\in[-\pi,\pi]$. Let $\{t_k\}_1^\infty$ be a set of discontinuity points of the first kind of the function $\alpha(\theta)$ on $(0,\pi)$, and let it have an only limit point $t_0 \in (0,\pi)$. Assume that the function $\alpha(\theta)$ has finite left and right limits at the point t_0. Moreover, let

$$\sum_{k=1}^{\infty} \left| \alpha(t_k+0) - \alpha(t_k-0) \right| < +\infty . \tag{13}$$

Assume

$$\frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi} \neq -\frac{1}{p(t_i)} + k, \quad i=\overline{1,\infty}, \tag{14}$$

for every integer k. Let

$$\frac{\pi}{2p(0)} + \left(n_0 - \frac{1}{2}\right)\pi < \alpha(0) < \frac{\pi}{2p(0)} + n_0\pi \tag{15}$$

for some integer n_0. Denote by r the number, after which the following conditions hold:

$$-\frac{\pi}{p(t_k)} < \alpha(t_k-0)-\alpha(t_k+0) < \frac{\pi}{q(t_k)},$$

$k=\overline{r,\infty}$. Renumber the elements of the set $\{t_i\}$, $i=\overline{1,r}$, in an ascending order and denote it by $\{t_i\}_1^r$, $0<t_1<\ldots<t_r<\pi$. Define the integers n_i, $i=\overline{1,r}$, from the following conditions:

$$-\frac{1}{p(t_i)} < \frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi} + n_i - n_{i-1} < \frac{1}{q(t_i)}, \quad i=\overline{1,r} . \tag{16}$$

Theorem 4. *Let $\alpha(t)$ be a real, piecewise continuous, odd function on $[-\pi,\pi]$, and let the conditions (13)-(15) hold with respect to the discontinuities of $\alpha(t)$. The integers n_i, $i=\overline{1,r}$, are defined by the conditions (15) and (16). Also let*

$$\alpha(\pi) \neq -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi .$$

Then, for the exponential system (12) to form a basis for the space $L_{p(\cdot)}(-\pi,\pi)$, it is sufficient that the inequality

$$-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+(n_r+1)\pi$$

holds. Besides, if

$$\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi\ ,$$

then the system (12) is not complete, but minimal in $L_{p(\cdot)}(-\pi,\pi)$. *And, if*

$$\alpha(\pi)\geq-\frac{\pi}{2p(\pi)}+(n_r+1)\pi\ ,$$

then this system is complete, but not minimal in $L_{p(\cdot)}(-\pi,\pi)$.

In **2.3**, a perturbed cosine system with a piecewise continuous phase is considered. Sufficient conditions on the jumps of the phase are found which provide that this system forms a basis for the generalized Lebesgue spaces.

In **2.4**, the validity of Riesz theorem on the functions from the Hardy class with respect to the arbitrary measurable subset of the unit circumference is proved.

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CONCLUSIONS

This thesis is dedicated to the study of basis properties of single exponential system with complex-valued coefficients, perturbed exponential system and cosine system in the Lebesgue spaces with variable summability index.

The following main results are obtained in this work:

- homogeneous Riemann problem is considered in the case where the coefficient of the problem satisfies some condition on a semicircumference and the general solution of this problem is constructed;

- nonhomogeneous Riemann problem is considered in the case where the coefficient of the problem and its right-hand side satisfy some conditions on a semicircumference and the sufficient condition is found which provides the solvability of this problem in the Hardy classes with variable summability index;

- single exponential system (1) with the complex-valued coefficients $a(t)$, $b(t)$ is considered in the case where the arguments $\arg a(t)$ and $\arg b(t)$ may have an infinite number of discontinuity points of the first kind on $[0,\pi]$ and the sufficient and necessary condition on the jumps of the function $\arg a(t) - \arg b(t)$ is found which provides that this system forms a basis for $L_{p(\cdot)}(0,\pi)$;

- exponential system (2) with a piecewise continuous phase $\gamma(\cdot)$, which is odd on $[-\pi,\pi]$ and may have an infinite number of discontinuity points of the first kind, is considered on $[-\pi,\pi]$ and the sufficient condition on the jumps of the function $\gamma(\cdot)$ is found which provides that this system forms a basis for $L_{p(\cdot)}(-\pi,\pi)$;

- system of cosines (3) with a piecewise continuous phase $\gamma(\cdot)$ is considered and, using the results obtained for the exponential system (2), the sufficient and necessary condition for the basicity of (3) in $L_{p(\cdot)}(0,\pi)$ is found.

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1. Наджафов, Т.И, Алескеров, М.И. Об одной задаче Римана в обобщенных классах Харди // Нахчыванский Государственный Университет, Научные труды, сер. физ. мат и техн. наук. №9 (65), -2015. с. 3-12.
2. Najafov, T.I., Aleskerov, M.I. On a Riemann problem in a generalized Hardy classes // 7-th International Conference on "Mathematical Analysis, Differential Equations and Their Applications", MADEA -7, -Baku: - 08-13 September, -2015, -p. 126
3. Mirzoyev, V.S., Aleskerov, M.I. On a special nonhomogeneous Riemann problem in generalized Hardy classes.// International Journal of Mathematical Analysis -2016. v. 10, № 20, pp. 965 – 979
4. Quliyeva, A.A., Alasgarov, M. On the solvability of the homogeneous Riemann problem // International Workshop on "Non-harmonic Analysis and Differential Operators, - Baku: -25-27 May, -2016, -pp. 93-94
5. Bilalov, B.T., Huseynli, A.A., Aleskerov, M.I. On the basicity of unitary system of exponents in the variable exponent Lebesgue spaces // -Baku: Transactions of NAS of Azerbaijan, issue Mathematics, -2017. v. XXXVII, № 1, - pp. 1-14
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7. Касумов, З.А., Алескеров, М.И. О базисности возмущенной системы косинусов в обобщенных пространствах Лебега // “Operators, Functions, and Systems of Mathematical Physics” Conference International conference, -Baku: Khazar university, -10-14 June, -2019, -p.167-168.
8. Aleskerov, M.I. An analogue of the Riesz theorem in Hardy-Morrey classes // -Baku: Transactions of ANAS. Series of Physical - Technical and Mathematical Sciences, -2020. v. 40, №1, -pp. 54-60

9. Aleskerov, M.I. On basicity of a perturbed system of cosines with unit in generalized Lebesgue spaces // Journal of Contemporary Applied Mathematics, -2020., v. 10, № 1, - pp. 46-57.
10. Aleskerov, M.I. On basis properties of a perturbed system of cosines in generalized Lebesgue spaces // 4th International E-Conference on “Mathematical Advances and its Applications”, -Istanbul,- Turkey: -26-29 May, -2021, -p. 144
11. Aleskerov, M.I. An analogue of the Riesz theorem in Hardy-Morrey classes // 4th International E-Conference on Mathematical Advances and its Applications, -Istanbul, Turkey: -26-29 May,-2021, -p. 145.

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