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ABSTRACT

of the dissertation for the degree of Doctor of philosopy

EMBEDDING THEOREMS IN SOBOLEV-MORREY TYPE SPACES

Specialty: 1202.01 – Analysis and functional analysis

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and the degree of development of the topic.

Many problems related to construction of a space of differentiable functions with many group of variables, study of their properties, the proof of some integral inequalities of embedding type theorems in these spaces belong to the section of “space theory” of mathematical analysis. Along with free development, from the point of view of function theory “embedding theory” is a section with efficient applications to theory of partial differential equations. Developments of functional analysis and reguirements of theory of partial differential equations led to the investigation of new type functional spaces. Embedding theory of the space of differentiable functions with many variables related to the solution of a number of problems of mathematical physics was first studied in S.L.Sobolev’s works [39,40]. In his work S.L.Sobolev has introduced isotropic $W_p^{(l)}(G)$ space consisting of the set of functions determined in $n-$dimensional domain $G \subset R^n$ and with the finite norm

$$\sum_{|\alpha| \le l} \left\| D^{\alpha} f \right\|_{L_p(G)} (l \in N, \quad p \ge 1, \ \alpha = (\alpha_1, ..., \alpha_n),$$

$$\alpha_j \ge 0 - \text{are integers}, \ j = 1,2, \ldots, n)$$

and proved several embedding theorems by means of integral representation from this space and described by its own derivatives. In other works, S.L.Sobolev has proved some integral inequalities for the functions from the isotropic space introduced by him and has given successful applications to the study of a class of mathematical physics problems, more exactly to the study of boundary value problems stated for elliptic and hyperbolic equations.

Later, theory of embedding in function spaces was studied and developed by many mathematicians, including S.M.Nikolski, V.P.Il’in, O.V.Besov, L.D.Kudryatsev, P.I.Lizorkin, A.J.Jabrayilov, T.I.Amanov, S.V.Uspenski, H.Triebel, V.Mazya, A.S.Jafarov, V.I.Burenkov, V.S.Guliyev, M.S.Jabrayilov, R.M.Rzayev and others.

Starting from the early sixties of the XX century, the research of a class of partial differential equatipns made necessary to study function spaces with dominant mixed derivatives. This class of functions can be studied in the previously applied classic $W_p^l, H_p^l, B_{p,\theta}^l, F_{p,\theta}^l$ spaces, but this time we would have required higher order derivatives from the solution in advance.

Such spaces were first studied S.M.Nikolsky, and then were developed by T.I.Amanov, A.J.Jabrailov, S.O.Ualiyev, A.M.Najafov and others. Sobolev, Nikolskii, Besov and Lizorkin-Tribel spaces were denoted by the authors as $S_p^l W$, $S_p^l H$, $S_{p,\theta}^l B$ and $S_{p,\theta}^l F$, respectively.

In those years, at first in V.P.Il'ins and later A.J.Jabrailov, A.M.Najafov, T.A.Shubochkina, L.Sh.Gadimova, A.T.Orujova and other's papers another type spaces, i.e. the spaces corresponding to the case when vectors with "smoothing exponents" are not on the coordinate axis of vectors, but on the positive half-planes were studied. Starting from the 90-s of the last century, for the Jacobian of the functions from the space $W_{loc}^{1,p}(R^n)$ to be local integrable, T.Ivaniec and S.Sbordonen began to study spaces being a modification of Lebesgue space and afterwards called a grand Lebesgue space and denoted as $L_{p)}$.

Later, such spaces were studied and developed by many mathematicians including L.Greco, A.Fiorenza, G.E.Karadzov, S.G.Samko, V.M.Kokilashvili, A.Meskhi, H.Rafeiro, A.Gogatishvili, S.M.Umarkhadzhiev, A.M.Najafov, M.R.Formica and others.

Starting with thirties of the past century, in order to study the smoothness of the solution of partial differential equations, a parametric space of many variable differential functions began to be studied.

Parametric spaces were first studied by Ch.Morrey (afterwards called Morrey spaces), and then were studied and developed by S.Kampanato, V.P.Il'in, I.Ross, Y.V.Netrusov, V.S.Guliyev, V.I.Burenkov, A.M.Najafov, H.V.Guliyev, A.Gogatishvili, R.Ch.Mustafayev, B.T.Bilalov, M.Taylor, Kh.Zhon, G.D.Fazio, M.Ragua, Y.Sawano, D.Fan, S.Lu, D.Yang, Y.Giga, T.Miyakama, M.A.Ragusa, L.Tang, J.Xu, G.Stampachia, A.Mazzucato, J.J.Hasanov, M.Omarova, A.T.Orujova and others.

Note that the results obtained in Morrey-type spaces were successfully applied in the papers of R.V.Huseynov, A.Mazzucato, V.S.Guliyev, M.A.Ragusa, A.M.Najafov and others for studying existence, uniqueness and smoothness problems of a class of differential equations. It should be noted that when smoothness problems were studied by the above mentioned authors, it was proved that "smoothness" exponents were higher than the previous works.

The topic of the dissertation work was devoted to the study of parametric spaces of differentiable functions with many group of variables. More exactly, in the dissertation work the Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ whose all the vectors $l^i = (l_1^i, l_2^i, \ldots, l_n^i)$ $(i = 0,1,\ldots,n)$ with "smoothness" exponent are not simultaneously on $n-$dimensional surface, the Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$, the grand Sobolev-Morrey spaces $\bigcap_{i=0}^{n} L_{p^i),\chi,\alpha}^{<l^i>}(G)$, the Nikolskii-Morrey $\bigcap_{i=0}^{n} L_{p^i),\chi,\alpha}^{<l^i>}(G)$ space, the Sobolev-Morrey space $\bigcap_{i=0}^{n} \mathscr{L}_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$, whose number of "smoothness exponentss" is 2^n and also fractional order grand Sobolev-Morrey space $\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ and grand Sobolev-Morrey spaces $S_{p),\chi),a,\alpha}^{l} W(G)$ with dominant mixed derivatives are studied.

In other words, by means of the integral representation method, some differential and difference-differential properties of functions from these spaces are studied.

As effective applications of the results of the dissertation when studying a class of higher order partial differential equations shows is of great importance the relevance of the topic of the dissertation.

Object and subject of the research. Research objects and subjects of the dissertation work as the followings: embedding of a class of Sobolev-Morrey type spaces and studying some properties of functions from these spaces from the point of view of embedding theory.

The goal of the thesis is to construct new Morrey-type spaces and to study some properties of these functions from these spaces from the point of view of embedding theory.

Research methods. In the dissertation work, the method of integral representation of functions determined in $n-$dimensional domains, the methods of functional analysis, function theory and the methods of mathematical analysis were used.

The main hypothesis to be defended. The following hypotheses are to be defended:

- embedding of new Morrey type spaces;
- study some features of included spaces;
- to prove both embedding and interpolation type theorems in the included spaces by the integral representation method;
- to prove by the integral representation method that generalized mixed derivatives of functions from these spaces satisfy the generalized Hölder condition.

Scientific novelty of the research. The following main results were obtained:

- new Morrey spaces are introduce;
- some properties of indroduced spaces are studied;
- both embedding and interpolation type theorems are proved in the introduced spaces by the integral representation method.
- it is proved by the integral representation method that generalized mixed derivatives of functions from these spaces satisfy the Hölder condition.

Theoretical and practical value of the research.

The dissertation work is of theoretical character. The results obtained arouses scientific interest in functional spaces theory.

Furthermore, the results obtained in the work can be applied in studying the existence, uniqueness and smoothness of solution of boundary value problems stated for a class of higher order partial differential equations.

Approbation and application. The results of the dissertation work were reported in the seminars of the departments of “Mathematical analysis” (corr.-member of ANAS V.S.Guliyev) and "Functional theory" (doctor of math. sc. V.E.Ismayilov) of the Institute of Mathematics and Mechanics of ANAS and the department of "Higher Mathematics" of Azerbaijan Architecture and Construction University, International Conference “Operators in Morrey type spaces and applications” in 2017 and 2019, the International Conference “Theoretical and Applied Problems of Mathematics” (SSU 2017), “Operators, functions and systems of mathematical phystis” conference (KHAZAR Universty, 2018) and in the Scientific Conference “Priorities of education policy in Azerbaijan ”(MSU, 2017).

Author's personal contribution is to show the goal of the research and to select its direction. The obtained results, belong to the author.

Author's publications. In publications recommended by the Higher Attestation Commision under the President of the Republic of Azerbaijan 6 papers, 4 conference materials and 4 abstracts. 2 of them were published in the journal included into Web of Science scientometric database and 1 paper in a journal included in Scopus database.

The name of the institution where the dissertation work was performed. The work was performed in “Mathematical analysis” department of Ganja State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately).

The title page consists of 333 signs, contents - 1102 signs, the introduction – 46000 signs, I chapter – 96000, II chapter – 70000, conclusion- 510. The total volume of the dissertation consists of 213612 signs.

THE MAİN CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, 2 chapters and a list publications.

In chapter I at first we introduce generalized Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$, the generalized grand Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i),\chi,a}^{<l^i>}(G)$, fractional order grand Sobolev-Morrey $W_{p),\chi,a}^{l}(G)$ and generalized Nikolskii-Morrey space $\bigcap_{i=0}^{n} \mathscr{L}_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$. Then, by the integral representation method we study differential properties of fnctions from these spaces. It is proved that the functions from these spaces satisfy tghe generalized Holder condition.

Let G be a domain in R^n.

Definition 0.1. The generalized Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ consists the set of all locally summable functions f defined on G and having all weak mixed derivatives $D^{l^i} f \left(l^i \in N_0^n,\ i = 0,1,\ldots,n\right)$ on G and with finite norm

$$\|f\|_{\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G)} = \sum_{i=0}^{n} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta,G}.$$

Here

$$\|f\|_{p,\varphi,\beta,G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}} \left(\left|\varphi([t]_1)\right|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right),$$

$l^i = \left(l_1^i,\ldots,l_n^i\right)$, $l_j^0 \geq 0$, $l_j^i \geq 0$ $\left(j \neq i = 1,2,\ldots,n\right)$, $l_i^i > 0\left(i = 1,2,\ldots,n\right)$-are integers, and let $1 \leq p^i < \infty$, $i = 0,1,\ldots,n$;

$$\left|\varphi([t]_1)\right|^{-\beta}=\prod_{j=1}^{n}(\varphi_j([t]_1))^{-\beta_j};\ \beta_j\in[0,1],(j=1,2,\dots,n),[t]_1=\min\{1,t\}$$

and for each $x\in R^n$ and $t>0$

$$G_{\varphi(t)}(x)=G\cap I_{\varphi(t)}(x)=G\cap\left\{y:|y_j-x_j|<\frac{1}{2}\varphi_j(t),\ \ j=1,2,\dots,n\right\}.$$

Here $\varphi(t)=(\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t))$ is a vector-function, where the components functions φ_j are Lebesgue measurable functions satisfy conditions

$$\varphi_j(t)>0\ \ (t>0),\quad \lim_{t\to+0}\varphi_j(t)=0,\quad \lim_{t\to+\infty}\varphi_j(t)=K_j,$$
$$0<K_j\le\infty\ ,j=1,2,\dots,n\ .$$

We will denote the set of such functions by N.

In particular, if $l^0=(0,\dots,0),\ l^i=(0,\dots,0,l_i,0,\dots,0),\ i=1,2,\dots,n;$

$p^i=p\,(i=0,1,\dots,n)$, then the space $\bigcap_{i=0}^{n}L^{<l^i>}_{p^i,\varphi,\beta}(G)$ and the space $W^{l}_{p,\varphi,\beta}(G)$ coincide.

Definition 0.2. The generalized Nikolskii-Morrey space $\bigcap_{i=0}^{n}\mathscr{L}^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ consists the set of locally summable functions f defined on G and with finite norm

$$\|f\|_{\bigcap\mathscr{L}^{<l^i>}_{p^i,\varphi,\beta}(G)}=\sum_{i=0}^{n}\sup_{0<t<t_0}\frac{\left\|\Delta^{m^i}(\varphi(t),G_{\varphi(t)})D^{k^i}f\right\|_{p^i,\varphi,\beta}}{\prod_{j=1}^{n}\varphi_j(t)^{l^i_j-k^i_j}}.$$

Here

$m^i \in N^n, \; k^i \in N_0^n, \; l^i = (l_1^i,...,l_n^i), \; l_j^0 \geq 0, \; l_j^i \geq 0, \; (i \neq j = 1,...,n),$

$l_i^i > 0, \; (i = 1,2,...,n)$ and t_0 be a fixed positive number,

$$\left\|\Delta^{m^i}(\varphi(t), G_{\varphi(t)})f\right\|_{p^i,\varphi,\beta} = \left\|\Delta^{m^i}(\varphi(t))f\right\|_{p^i,\varphi,\beta;G_{\varphi(t)}} = \left\|\Delta^{m^i}(\varphi(t))f\right\|_{L_{p^i,\varphi,\beta}(G_\varphi)},$$

and

$$\Delta^{m^i}(\varphi(t), G_{\varphi(t)})f(x) = \begin{cases} \Delta^{m^i}(\varphi(t))f(x), & [x, \; x + m^i\varphi(t)] \subset G, \\ 0, & [x, \; x + m^i\varphi(t)] \overline{\subset} G, \end{cases}$$

$$\Delta^{m^i}(\varphi(t))f(x) = \sum_{j=0}^{m^i}(-1)^{m^i-j} C_{m^i}^j f(x + j\varphi(t)).$$

Definition 0.3. The generalized grand Sobolev-Morrey space $\bigcap_{i=0}^{n} L_{p^i),\chi,a}^{<l^i>}(G)$ consists of the set of all locally summable functions f defined on $G \subset R^n$ with weak mixed derivatives $D^{l^i} f \quad (l^i \in N_0^n, \; i = 0,1,..., n)$ on G and determined by the finite norm

$$\|f\|_{\bigcap\limits_{i=0}^{n} L_{p^i),\chi,a}^{<l^i>}(G)} = \sum_{i=0}^{n}\left\|D^{l^i} f\right\|_{p^i),\chi,a;G},$$

where

$$\|f\|_{p^i),\chi,a;G} = \|f\|_{L_{p^i),\chi,a}(G)} =$$

$$= \sup_{\substack{0<t\leq d_0, \\ x\in G, \\ 0<\varepsilon<p-1}} \left(\frac{1}{t^{|\chi|a}} \frac{\varepsilon}{\left|G_{t\chi}(x)\right|} \int_{\left|G_{t\chi}(x)\right|} |f(y)|^{p-\varepsilon} \, dy \right)^{p-\varepsilon},$$

$d_0 = diam G; \; 1 < p^i < \infty; \; l^i = (l_1^i, l_2^i,..., l_n^i) \in N_0^n,$ more exactly, $l_j^0 \geq 0$ $(j = 1,2,...,n),$ $l_j^i \geq 0 \; (i \neq j = 1,2,...,n), \; l_l^i > 0 \; (i = 1,2,...,n)$ -are integers; $\chi = (\chi_1, \chi_2,..., \chi_n), \;\; \chi_j > 0 \; (j = 1,2,...,n); \; a \in [0,1]$.

Definition 0.4. The fractional order grand Sobolev-Morrey space $W^l_{p),\chi,a}(G)$ consists the set of all locally summable functions f defined on bounded domain $G \subset R^n$ and having all weak mixed derivatives $D^{l^i} f \left(l^i \in N_0^n,\ i = 0,1,\dots,n\right)$ on G and with finite norm

$$\|f\|_{W^l_{p),a,\chi}(G)} = \|f\|_{p),a,\chi;G} + \sum_{i=1}^{n} \left\| D_i^{l_i} f \right\|_{p),a,\chi;G},$$

where

$$\|f\|_{p),a,\chi,G} = \|f\|_{L_{p),a,\chi}(G)} = \sup_{\substack{0<t\le d,\\ x\in G,\\ 0<\varepsilon<p-1}} \left(\frac{1}{t^{|\chi|a}} \frac{\varepsilon}{\left|G_{t^\chi}(x)\right|} \int_{G_{t^\chi}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}},$$

$l \in (0,\infty)^n;\ 1 < p < \infty;\ a \in [0,1];\ \chi \in (0,\infty)^n;\ D_i^{l_i} f = D_i^{[l_i]} D_{+i}^{\{l_i\}} f$, $[l_i]$ is the integer part of the number l_i, $\{l_i\}$ is the fractional part of the number l_i and

$$G_{t^\chi}(x) = G \cap I_{t^\chi}(x) = G \cap \left\{ y : \left|y_j - x_j\right| < \frac{1}{2} t^{\chi_j},\ j = 1,2,\dots,n \right\}.$$

We say that $D_i^{\{l_i\}} f$ is a generalized Sobolev fractional derivative of f on G, if

$$\int_G f(x)\left(D_i^{[l_i]} D_{+i}^{\{l_i\}} \varphi\right)(x) dx = (-1)^{[l_i]} \int_G \varphi(x)\left(D_i^{[l_i]} D_{+i}^{\{l_i\}} f\right)(x) dx$$

for all $\varphi \in C_0^\infty(G)$.

Note that here in the domain the Riemann-Liouville $\{l_i\}$ $\left(0 < \{l_i\} < 1\right)$ fractional order ordinary derivatives $D_{+i}^{\{l_i\}} f$ and $D_{-i}^{\{l_i\}} f$ are defined as follows :

$$\left(D_{+i}^{\{l_i\}} f\right)(x) = \frac{1}{\Gamma\left(1 - \{l_i\}\right)} \frac{\partial}{\partial x_i} \int_{G^{(i)}} \frac{f\left(x_1,\dots,x_{i-1},s_i,x_{i+1},\dots x_n\right)}{\left(x_i - s_i\right)^{\{l_i\}}} ds_i,$$

$$\left(D_{-i}^{\{l_i\}} f\right)(x) = -\frac{1}{\Gamma(1-\{l_i\})}\frac{\partial}{\partial x_i}\int_{\overline{G}^{(i)}} \frac{f(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots x_n)}{(s_i - x_i)^{\{l_i\}}} ds_i,$$

where x is a interior point of domain G, $\Gamma(\alpha)$ is the Euler gamma function, the domain $G^{(i)}$ and $\overline{G}^{(i)}$ are determined as follows:

$$G^{(i)} = \{(x_1,\ldots,x_{i-1}s_i,x_{i+1},\ldots x_n) \in G : x_j = const(j \neq i); s_i < x_i\},$$

$$\overline{G}^{(i)} = \{(x_1,\ldots,x_{i-1}s_i,x_{i+1},\ldots x_n) \in G : x_j = const(j \neq i); x_i < s_i\}.$$

Definition 0.5. We say that an open set $G \subset R^n$ satisfy the condition (A), if for any $\theta \in (0,1]^n$, $T_j \in (0,\infty)$ and for any $x \in G$ there exist a vector-function

$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t_1),x),\ldots,\rho_n(\varphi_n(t_n),x)),\ 0 \le t_j \le T_j,\ \ j = 1,2,\ldots,n$

such that

1) For each $j = 1,2,\ldots,n$ the component functions $\rho_j(\varphi_j(t_j),x)$ are absolutely continuous and for almost everywhere $t_j \in [0,T_j]$ and for almost everywhere $u_j \in [0,T_j]$ $\left|\rho'_j(u_j,x)\right| < 1$, here $\rho'_j(u_j,x) = \frac{\partial}{\partial u_j}\left(\rho_j(u_j,x)\right)$;

2) For each $j = 1,2,\ldots,n$

$$\rho_j(0,x) = 0,\ x + V(x,\theta) =$$
$$x + \bigcup_{\substack{0 \le t_j \le T_j,\\ j=1,2,\ldots,n}} [\rho(\varphi(t),x) + \varphi(t)\theta I] \subset G,$$

In particular, if $t_j = t$ $(j = 1,2,\ldots,n)$, then we will called the set $x + V(x,\theta)$ a set satisfying a flexible "$\varphi -$horn" condition, if $\varphi(t) = t^{\lambda}$ $(t^{\lambda} = (t^{\lambda_1},t^{\lambda_2},\ldots,t^{\lambda_n}))$ $\theta_j = \theta^{\lambda_j}$ $(j = 1,\ldots,n)$ then the set $x + V(x,\theta) \equiv x + V(x,\lambda,\theta)$ will be the set satisfying the "flexible $\lambda -$horn" condition. The class of domain satisfying the "flexible $\lambda -$horn" condition was introduced by O.V.Besov.

Theorem 0.1. Let $G \subset R^n$ be a domain satisfying "flexible φ-horn" condition, $1 \le p^i \le p \le \infty$, $(i = 0,1,\dots,n)$, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \ge 0$ are integers $(j = 1,\dots,n)$ and $\nu_j \ge l_j^0$; $\nu_j \ge l_j^i$ $(j \ne i = 1,\dots,n)$,

$$Q_T^i = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{l_j^i - \nu_j - (1-\beta_j p^i)\left(\frac{1}{p^i} - \frac{1}{p}\right)} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt < \infty,$$

and let $f \in \bigcap_{i=0}^n L_{p^i,\varphi,\beta}^{<l^i>}(G)$. Then the following embedding holds

$$D^\nu : \bigcap_{i=0}^n L_{p^i,\varphi,\beta}^{<l^i>}(G) \to L_{p,\psi,\beta^1}(G).$$

In other words, for $f \in \bigcap_{i=0}^n L_{p^i,\varphi,\beta}^{<l^i>}(G)$ there exists weak mixed derivatives $D^\nu f$ and the following inequalities are valid

$$\left\| D^\nu f \right\|_{q,G} \le C_1 \sum_{i=0}^n | Q_T^i | \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G}$$

$$\left\| D^\nu f \right\|_{p,\psi,\beta^1;G} \le C_2 \left\| f \right\|_{\bigcap_{i=0}^n L_{p^i,\varphi,\beta}^{<l^i>}(G)}, \quad (p^i \le p < \infty).$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{l_j^i - \nu_j - (1-\beta_i p)\frac{1}{p^i}} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt < \infty, \quad i = 1,2,\dots,n.$$

Then the weak derivatives $D^\nu f(x)$ are continuous in the domain G and

$$\sup_{x \in G} | D^\nu f(x) | \le C_1 \sum_{j=1}^n | Q_{T,0}^i | \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta,G},$$

where $0 < T \le \min\{1, T_0\}$, T_0 is a fixed positive number; $\psi \in N$, $\varphi_j(t) \le \psi_j(t)$, $\beta_j^1 = \dfrac{\beta_j p}{p^i}$, $j = 1,2,\dots,n$, C_1 and C_2 are positive numbers independent of f. In additional, the constant C_1 is independent of T as well.

Theorem 0.2. Let $G \subset R^n$ be a bounded open set satisfying the "flexible λ-horn" condition and assume that, $1 < p^i < p \le \infty$, $|\chi| \le \frac{|\lambda|}{1+a}$; $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \ge 0$, $(j = 1,2,...,n)$ and $\nu_j \ge l_j^0$, $\nu_j \ge l_j^i$ $(j \ne i = 1,2,...,n)$ are integers such that

$$m^i = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\chi| - |\chi| a) \times$$
$$\times \left(\frac{1}{p^i - \varepsilon} - \frac{1}{p - \varepsilon} \right) > 0 \ (i = 1,2,...,n)$$

and let $f \in \bigcap_{i=0}^{n} L^{<l^i>}_{p^i), \chi, a}(G)$. Then the following embedding holds

$D^\nu : \bigcap_{i=0}^{n} L^{<l^i>}_{p^i), \chi, a}(G) \to L_{p-\varepsilon}(G)$, more exactly, for $f \in \bigcap_{i=0}^{n} L^{<l^i>}_{p^i), \chi, a}(G)$ there exist weak mixed derivatives $D^\nu f$ and

$$\left\| D^\nu f \right\|_{p-\varepsilon, G} \le C^1(\varepsilon) \sum_{i=0}^{n} T^{m^i} \left\| D^{l^i} f \right\|_{p^i), \chi, a; G}.$$

In particular, if

$$m^{i,0} = (l^i, \lambda) - (\nu, \lambda) - (|\lambda| - |\chi| - |\chi| a) \frac{1}{p^i - \varepsilon} > 0, \quad (i = 1,2,....,n),$$

then the weak derivatives $D^\nu f$ are continuous in the domain G and

$$\sup_{x \in G} \left| D^\nu f(x) \right| \le C^1(\varepsilon) \sum_{i=0}^{n} \prod_{j \in e_n} T^{m^i, 0} \left\| D^{l^i} f \right\|_{p^i), \chi, a; G},$$

where $0 < T \le d_0$, $C^1(\varepsilon) = C^1 \varepsilon^{-\frac{1}{p^i - \varepsilon}}$; the constant $C^1 = \max\{C^0, C^1, \dots, C^n\}$ is a positive number independent of f, T and ε.

In chapter I it is also proved that the generalized mixed derivatives of functions from generalized Sobolev-Morrey and grand

Sobolev-Morrey spaces within the conditions of Theorem 0.1 and Theorem 0.2, satisfy the Hölder condition. We note that the "smoothness index" for the grand Sobolev-Morrey type is higher than "smoothness exponents" in the previous works.

Theorem 0.3. Assume that $G \subset R^n$ satisfies the "flexible φ-horn"condition $1 \le p^i \le p \le \infty$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \ge 0$ $j = 1,2,...,n$ are integers, $\nu_j \ge l_j^0$, $\nu_j \ge l_j^i$ $(j \ne i = 1,2,...,n)$, $\nu_i < l_i^i$ $(i = 1,2,...,n)$; $Q_T^i < \infty$ $(i = 1,2,....,n)$ and let $f \in \bigcap_{i=0}^{n} \mathscr{L}^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$. Then the following embedding is valid $D^\nu : \bigcap_{i=0}^{n} \mathscr{L}^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi) \to \mathscr{L}_{q,\psi,\beta^1}(G)$, in other words, for $f \in \bigcap_{i=0}^{n} \mathscr{L}^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ there exist weak mixed derivatives $D^\nu f$ and the following inequalities hold

$$\left\| D^\nu f \right\|_{p,G} \le C_1 \sum_{i=0}^{n} | H_T^i | \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_j(t))^{-l_j^i} \Delta^{m^i}(\varphi\ (t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta}$$

$$\left\| D^\nu f \right\|_{p,\psi,\beta^1;G} \le C_2 \left\| f \right\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)}, \quad p^i \le p < \infty.$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \prod_{j \in e_{l^i}} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_j^i}} dt < \infty,$$

then the weak derivatives $D^\nu f(x)$ are continuous in the domain G and

$$\sup_{x \in G} | D^\nu f(x) | \le$$

$$\le C_1 \sum_{i=1}^{n} | H_{T,0}^i | \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_j(t))^{-l_j^i} \Delta^{m^i}(\varphi\ (t), G_{\varphi(t)}) f \right\|_{p^i,\varphi,\beta}.$$

Here

$$H_T^i = \begin{cases} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j - \left(1-\beta_j p^0\right)\left(\frac{1}{p^0} - \frac{1}{p}\right)}, & i = 0 \\ Q_T^i & i = 1,2,\dots,n \end{cases}$$

$$\left(H_{T,0}^i = \begin{cases} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j - \left(1-\beta_j p^0\right)\frac{1}{p^0}}, & i = 0 \\ Q_{T,0}^i & i \neq 1,2,\dots,n \end{cases} \right)$$

and $0 < T \le \min\{1, T_{0j}\}$, T_0 is a fixed positive number the constants, C_1, C_2 are positive numbers independent of f, C_1 and also of T.

Theorem 0.4. Assume that all the conditions of Theorem 0.3 are satisfied. Then weak mixed derivatives $D^\nu f$ undert the conditions $Q_T^i < \infty$ $(i = 1,2,\dots,n)$ satisfies the generalized Hölder condition in the domain G, i.e. the following inequalities are valid:

$$\left\|\Delta(\gamma, G) D^\nu f\right\|_{p,G} \le C \|f\|_{\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)} \, | H(|\gamma|, \varphi; T) |,$$

where the constant C is a positive numbers independent of $f, |\gamma|$ and T.

In particular, if $Q_{T,o}^i < \infty$ $i = 1,2,\dots,n$, then

$$\sup_{x \in G} \left|\Delta(\gamma, G) D^\nu f(x)\right| \le C \|f\|_{\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)} \, | H_0(|\gamma|, \varphi; T) |,$$

where

$$| H(|\gamma|, \varphi : T) | = \max_i \{ |\gamma|, Q'^i_{|\gamma|}, Q'^i_{|\gamma|,T} \}$$

$$\left(| H_0(|\gamma|, \varphi : T) | = \max_i \{ |\gamma|, Q'_{|\gamma|,0}, Q'_{|\gamma|,T,0} \} \right)$$

Theorem 0.5. Assume that the domain G is a bounded open set satisfying the "flexible λ $(\lambda \in 0,\infty)^n$) -horn" in R^n and let

$$1 < p < q \le \infty;\ 0 \le |\chi| \le \frac{|\lambda|}{1+a};\ \nu = (\nu_1, \nu_2, \ldots, \nu_n),$$

$\nu_j \ge 0\ (j = 1,2,\ldots,n)$; $\overline{\mu_i} > 0\ (i = 1,2,\ldots,n)$ and let $f \in W^l_{p),a.\chi}(G)$.

Then for each $0 < \varepsilon < p-1$, an embedding $D^\nu : W^l_{p),a.\chi}(G) \to L_{q-\varepsilon}(G)$ is valid. In other words, the following inequality holds

$$\left\| D^\nu f \right\|_{q-\varepsilon,G} \le C(\varepsilon)\left(T^{\overline{\mu_0}} \|f\|_{p),a,\chi;G} + \sum_{i=1}^n T^{\overline{\mu_i}} \left\| D_i^{l_i} f \right\|_{p),a,\chi;G} \right),$$

where $\overline{\mu_0} = \overline{\mu_i} - \lambda_i l_i$.

In particular, if

$$\overline{\mu_{i,0}} = \lambda_i l_i - |\nu,\lambda| - \left(|\lambda| - |\chi| - |\chi| a\right)\frac{1}{p-\varepsilon} > 0 \quad (i = 1,2,\ldots,n),$$

then $D^\nu f(x)$ in continuously in the domain G and

$$\sup_{x\in G}\left| D^\nu f(x) \right| \le C(\varepsilon)\left(T^{\overline{\mu_{0,0}}} \|f\|_{p),a,\chi;G} + \sum_{i=1}^n T^{\overline{\mu_{i,0}}} \left\| D_i^{l_i} f \right\|_{p),a,\chi;G} \right),$$

where $0 < T \le \min\{T_0, 1\}, T_0$, T_0 is a fixed positive number; $C(\varepsilon) = C\varepsilon^{-\frac{1}{p-\varepsilon}}$ and C is a positive constant independent of f, T and ε. Moreover, we proved a compactness of embeddings of grand-Sobolev space $W^l_{p),\chi,a}(G)$ and show that mixed derivatives of functions in this space satisfy the Hölder condition.

In chapter II we study the Sobolev-Morrey space $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ with 2^n number "smoothness exponents" and grand Sobolev-Morrey spaces $S^l_{p),\chi),a,\alpha}W(G)$ with dominant mixed derivatives. More exactly, at first, the spaces are constructed, and then Sobolev type inequalities for the weak mixed derivatives of functions from these spaces is proved.

At the end, some theorems on differential and difference-differential properties of the functions belonging to intersections of the generalized Nikolskii type spaces studied from embedding point of view.

Definition 0.6. The set of functions f defined on the domain $G\subset R^n$ and having locally summable weak mixed derivatives $D^{l^i} f \quad (i=1,2,\ldots,2^n)$ and finite norm

$$\|f\|_{\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} = \sum_{i=1}^{2^n} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G}$$

is called a generalized Sobolev-Morrey space $\bigcap_{i=1}^{2n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ with 2^n number “smoothness exponents” . Here

$$\|f\|_{p^i,\varphi,\beta,G} = \|f\|_{L_{p^i,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t_j>0,\\ j=1,2,\ldots,n}} \left(\prod_{j=1}^{n} (\varphi_j([t_j]_1))^{-\beta_j} \|f\|_{p^i,G_{\varphi(t)}(x)} \right),$$

$1\le p^i<\infty,\ l^i=(l_1^i,l_2^i,\ldots,l_n^i),\ \ i=1,2,\ldots,2^n,\ e_n=\{1,2,\ldots,n\},\ e^i\subseteq e_n$ (the number of all possible vectors e^i is 2^n), $l_j^i\in N,\ j\in e^i$, $l_j^i\in N_0,\ j\in e^n\setminus e^i;\ \beta_j\in[0,1],\ j\in e_n;$

$\varphi(t)=(\varphi_1(t_1),\varphi_2(t_2),\ldots,\varphi_n(t_n))$ be a vector function such that $\varphi_j(t_j)>0\ (t_j>0)$, $\lim\limits_{t_j\to+0}\varphi_j(t_j)=0$, $\lim\limits_{t_j\to+\infty}\varphi_j(t_j)=K_j$, $0<K_j\le\infty,\ \ [t_j]_1=\min\{1,t_j\},\ \ j\in e_n$ and components of vector function are Lebesgue measurable functions.

Definition 0.7. The set of functions f defined in a bounded domain $G\subset R^n$ having locally summable weak mixed derivatives $D^{l^e} f \quad (l^e=(l_1^e,l_2^e,\ldots,l_n^e),\ l_j^e\in N,\ \ j\in e\subseteq e_n)$ and finite norm

$$\|f\|_{S_{p),\chi),a,\alpha}W(G)} = \sum_{e\subseteq e_n} \left\| D^{l^e} f \right\|_{p),\chi),a,\alpha;G},$$

is called a grand grand Sobolev-Morrey space $S^l_{p),\chi),a,\alpha}W(G)$ with dominant mixed derivatives. Here

$$\|f\|_{p),\chi),a,\alpha;G} == \sup_{\substack{x\in G \\ 0<t_j\le d_j \\ 0<\varepsilon<s_m}} \left(\frac{1}{\prod_{j\in e_n} t_j^{\chi_j a-\alpha_j\varepsilon}} \frac{\varepsilon}{\left|G^{(x)}_{t^\chi}\right|} \int_{G_{t^\chi}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}},$$

and $d_j \quad (j\in e_n)$ are diagonals of a n-dimensional rectangular parallelepiped $I_{t^\chi}(x)$; $1<p<\infty$; $s_m = \min\{s_1,\ldots,s_m\}$, $s_j = \min\left\{p-1, \frac{\chi_j a}{\alpha_j}\right\}$ $\alpha_j \ge 0 \quad (j\in e_n)$; $\chi\in(0,\infty)^n$; and $a\in[0,1]$ (we adopt the convention that $\frac{0}{0}=0$).

At first, in the case when the domain $G\subset R^n$ satisfy the condition (A), for the function from the space $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ we prove the embedding $D^\nu : \bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \to L_{p^i,\varphi,\beta^1}(G)$ and that the weak derivatives of the functions from this space satisfy the Hölder condition.

At the end for the functions from the grand grand Sobolev-Morrey space $S^l_{p),\chi),a,\alpha}W(G)$ we prove the following theorems.

Theorem 0.6. Let $G\subset R^n$ be a bounded domain satisfying condition (A) and let $1<p<q\le\infty$, $\chi_j \le \frac{1}{1+a}$, $\nu=(\nu_1,\ldots,\nu_n)$, $\nu_j \ge 0, (j\in e_n)$-are integers and

$$\mu_j = l_j - \nu_j - (1-\chi_j - \chi_j a_j + \alpha_j \varepsilon)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right) > 0 \quad (j \in e_n).$$

Then the following embedding holds

$$D: S^l_{p),\chi),a,\alpha} W(G) \to L_{q-\varepsilon}(G) \quad (0 < \varepsilon < s_m),$$

i.e. there exist weak mixed derivatives $D^\nu f$ of $f \in S^l_{p),\chi),a,\alpha} W(G)$ and

$$\left\| D^\nu f \right\|_{q-\varepsilon, G} \le C(\varepsilon) \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j}} \left\| D^{l^e} f \right\|_{p),\chi),a,\alpha; G}$$

Here

$$s_{e,j} = \begin{cases} \mu_j, \ j \in e \\ \nu_j - (1-\chi_j - \chi_j a + \alpha_j \varepsilon)\left(\dfrac{1}{p-\varepsilon} - \dfrac{1}{q-\varepsilon}\right), \quad j \in e' = e_n \setminus e. \end{cases}$$

In particular, if

$$\mu_{j,0} = l_j - \nu_j - (1-\chi_j - \chi_j a + \alpha_j \varepsilon)\frac{1}{p-\varepsilon} > 0 \quad (j \in e_n),$$

then weak mixed derivatives $D^\nu f$ are continuous in the domain G, and

$$\sup_{x \in G} \left| D^\nu f(x) \right| \le C(\varepsilon) \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j,0}} \left\| D^{l_e} f \right\|_{p),\chi),a,\alpha; G},$$

where $0 < T_j \le d_j \quad (j \in e_n), C(\varepsilon)$ is a constant independent of f and $T = (T_1, \ldots, T_n)$.

Theorem 0.7. Assume that all the conditions of theorem 0.6 are satisfied. Then if $\mu_j > 0 \ (j \in e_n)$, then the weak mixed derivatives $D^\nu f$ satisfy the Hölder condition with the metrics in $L_{q-\varepsilon}(G)$ by the exponent σ_j, more exactly, we have

$$\left\| \Delta(\xi, G) D^\nu f \right\|_{q-\varepsilon, G} \le C(\varepsilon) \| f \|_{S^l_{p),\chi),a,\alpha} W(G)} \prod_{j \in e_n} \left| \xi_j \right|^{\sigma_j},$$

where $\sigma_j \quad (j \in e_n)$ the numbers satisfying following conditions

$0 \le \sigma_j \le 1 \quad \text{if} \quad \mu_j > 1, \quad j \in e,$

$0 \le \sigma_j < 1 \quad \text{if} \quad \mu_j = 1, \quad j \in e; \qquad 0 \le \sigma_j \le 1\,, \quad j \in e' = e_n \setminus e\,,$

$0 \le \sigma_j \le \mu_j \quad \text{if} \quad \mu_j < 1, \quad j \in e;$

In particular, if $\mu_{j,0} > 0 \ \ (j \in e_n)$ then

$$\sup_{x \in G} \left| \Delta(\xi, G) D^{\nu} f(x) \right| \le C(\varepsilon) \|f\|_{S^l_{p),\chi),a,\alpha} W(G)} \prod_{j \in e_n} \left| \xi_j \right|^{\sigma_{j,0}} \ ,$$

where $\sigma_{j,0}$ satisfies the same conditions instead of σ_j, but instead of μ_j we take $\mu_{j,0}$.

In this chapter, we prove interpolation type theorems for Nikolskii-Morrey spaces consists the functions having weak mixed derivatives as well.

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CONCLUSIONS

In the dissertation work, at first, the spaces of differentiable functions with many group of variables with parameters is introduced. Then, by the integral representation method, both differential and difference-differential properties of functions from these space are studied.

The following results were obtained:

- new Morrey spaces were introduced;
- both embedding and interpolation type theorems are proved in the introduced spaces.
- it was proved that the weak mixed derivatives of the functions from this spaces satisfy the generalized Hölder condition.

The main results of the dissertation were published in the following works:

1. Babayev, R.F. The embedding theorems in generalized Sobolev-Morrey type space// "Riyaziyyatın nəzəri və tətbiqi problemləri" beynəlxalq elmi konfransın materialları, -Sumqayıt: -25-26 May, -2017, -s.30-31.
2. Najafov, A.M., Babayev, R.F. Some properties of Functions from generalized Sobolev-Morrey type spaces.//Mathematica Aeterna, -2017. v.7, № 3, -p. 301-311.
3. Babayev, R.F. Qarışıq törəmələri dominant olan ümumiləşmiş Sobolev-Morri fəzalarında daxilolma teoremləri // Azərbaycanda Təhsil Siyasətinin Prioritetləri: Müasir yanaşmalar elmi konfransın mater. -Mingəçevir: -15-16 Dekabr, -2017, -s.77-78
4. Babayev, R.F. The embedding theorems in generalized Sobolev-Morrey type space with dominanat mixed derivatives // "Operators, functıons and systems of mathematıcal physıcs" conference, - 21-24 May -2018, –Baku: Khazar University, -p.126.
5. Babayev, R.F. Some differential properties of generalized Nikolskii-Morrey type spaces // -Baku: Caspian Journal of Applied Mathematics, Ecology and Economics, -2018. v. 6, № 2, -p. 110-119.
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7. Babayev, R.F. Poincare type inequality in generalized Sobolev-Morrey type spaces. // Kutahya Dumlupinar University, OMTSA-2019, -Turkey: -p.27.
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10. Najafov, A.M., Babayev, R.F. On some differential properties of grand Sobolev-Morrey spaces of fractional order // Uzb. Math. Jour. -2021. v.65, Issue 2, -p.128-139

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