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ABSTRACT

of the dissertation for the degree of Doctor of Science

FRAME PROPERTIES OF DEGENERATE TRIGONOMETRIC SYSTEMS, BASIS PROPERTIES OF KOSTYUCHENKO TYPE SYSTEMS AND SOME APPLICATIONS

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GENERAL CHARACTERISTICS OF THIS WORK

Rationale of the work. Basis theory, as an independent research area, has been studied in numerous researches. The topics of these researches include, in particular, identification of criteria for basicity of a system in different spaces, investigation of basicity of system that are close to a bases in some sense, study of some properties of operators acting in spaces with a bases, researches on abstract conditions for a space to possess a bases and etc. As an example of scientific publications dedicated to the theory of bases we mention the monographs by I.Singer, R.Young, A.M.Sedletskii, C.Heil, B.T.Bilalov and etc, and overview papers by V.D.Milman, N.K.Bari, I.Gokhberg and A.Markus, A.M.Sedletskii and ets.

The information presented above describing only a very small part of the overall picture is intended to show that the inner theoretical issues of the theory of bases in its own is also of a great interest. However, applications of this theory to the solution of problems arising in other fields of mathematics and other natural sciences shows the importance of this theory also from the application point of view.

It should be noted that usually in the internal problems of the theory of bases, the basicity properties of systems that satisfy certain conditions that are not directly related to their form are considered; but in problems of an applied nature, it becomes necessary to study the basis property of systems of a certain form in certain specific spaces. In these problems, the form of the system under consideration varies depending on the problem that generates the system. As an example of such problems of an application character, the need of investigation of basicity of systems arising in the justification of the Fourier method, widely used in the theory of differential equations, can be shown.

In connection with applications in this or other problems, basicity properties of classical exponential $\left\{e^{int}\right\}_{n\in Z}$, sine $\left\{\sin nt\right\}_{n\in N}$ and cosine $\left\{\cos nt\right\}_{n\in Z^+}$ systems in different spaces have been widely studied. However, some problems of mathematics of a theoretical character and problems arising in applications require the investigation of basicity of not only these systems but also basicity of various modifications of these systems. As an example of such a theoretical problem, one can cite the result of K.I.Babenko: Babenko gave an example

$$\{|t|^{\alpha} \cdot e^{int}\}_{n\in Z},$$

where $|\alpha|<\frac{1}{2}$ and $\alpha\neq 0$, answering in the affirmative a question of N.Bari on the existence of normalized basis for $L_2(-\pi,\pi)$ that is not a Riesz basis;

Starting with this result, many papers were also devoted to the study of the basis properties (completeness, minimality, and Schauder basis property) of systems of the form

$$\left\{\prod_{j=1}^{r}\left|t-t_j\right|^{\alpha_j} e^{\text{int}}\right\}_{n\in Z},$$

$$\left\{\prod_{j=1}^{r}\left|t-t_j\right|^{\alpha_j} \cos nt\right\}_{n\in Z_+},$$

$$\left\{\prod_{j=1}^{r}\left|t-t_j\right|^{\alpha_j} \sin nt\right\}_{n\in N}.$$

For example, systems of the above type have been studied in connection with the study of the basis properties of eigenfunctions of certain differential operators in certain weighted spaces.

The mentioned result of Babenko was then extended by V.F. Gaposhkin in his famous work, where, in particular, a certain sufficient condition was given (on the weight function $\omega(t)$) for the system

$$\{\omega(t)e^{int}\}_{n\in Z}$$

to be a basis in $L_2(-\pi,\pi)$. Finally, a necessary and sufficient condition was obtained in terms of the weight function $\omega(t)$, which ensures the Schauder basicity of the exponential system $\{e^{int}\}_{n\in Z}$ in the weighted Lebesgue space $L_{p,\omega()}(-\pi,\pi)$; such a condition is the Mackenhaupt condition with respect to the weight function $\omega(t)$:

$$\sup_I \left(\frac{1}{|I|}\int_I \omega(t)dt\right)\left(\frac{1}{|I|}\int_I \omega^{-\frac{1}{p-1}}(t)dt\right)^{p-1} < \infty,$$

where sup is taken over all intervals $I \subset [-\pi,\pi]$, and $|I|$ denotes the length of the interval I. As an example of research in this direction, one can cite papers by the authors V.F. Gaposhkin, R. Hunt, B. Mackenhaupt, R. Wheeden, W.S. Young and others.

Note that the study of the basis properties of a system in weighted Lebesgue spaces $L_{p,\omega()}$ is equivalent to the study of similar properties of this system with the corresponding degenerate coefficient in the "ordinary" Lebesgue space L_p. Consequently, this criterion can also be considered as a necessary and sufficient condition for the Schauder basicity in L_p of an exponential system with degenerate coefficients $\{\omega(t)e^{int}\}_{n\in Z}$.

As an example of application of the theory of bases in other fields of mathematics, the need of investigation of basicity of systems arising in the spectral theory of differential operators can be shown. For example, the need of investigation of basicity of the so called Kostyuchenko system

$$S_\alpha^+ \equiv \{e^{i\alpha nt}\sin nt\}_{n\in \mathrm{N}},$$

where $\alpha \in \mathbf{C}$ is, in general, a complex number, arises in connection with the spectral problem for nonselfadjoint quadratic pencil

$$-y''(t)+2\alpha\lambda y'(t)+(\alpha^2+1)\lambda^2 y(t)=0, t\in(0,\pi)$$

with conditions $y(0)=y(\pi)=0$.

The problem posted in 1969 by A.G. Kostyuchenko suggesting to use only purely functional methods for the investigation of the completeness and basicity of the system S_α^+ in $L_2(0,\pi)$ is well known in the spectral theory of differential operator bundles. The first result in this direction was obtained in 1971 by B.Ya.Levin: he proved that the system S_α^+ is complete in the space $L_2(0,\pi)$ for all $\alpha \in iR$. It should also be noted that this result was also obtained before by M.G.Djavadov by using the different approach.

Beginning from the Djavadov's paper many papers have been dedicated to the investigation of basis properties (basicity, completeness, minimality) of the system S_α^+ in $L_p(0,\pi)$, $1 \le p < +\infty$, spaces. Some of the results obtained in this direction are as follows: in the case when

$$\alpha \in C \setminus \{(-\infty,-1) \cup (1,+\infty)\}$$

the criteria is obtained for the completeness and minimality of the system S_α^+ in $L_2(0,\pi)$. In particular, it is complete and minimal in $L_2(0,\pi)$ for $\forall \alpha \in iR$, i.e. for all imaginary α.

In general, completeness and minimality of the mentioned system have been studied in many papers, and it can be said that the completeness and minimality of this system were studied almost completely. However, investigation of the Shauder basicity of the Kostyuchenko system S_α^+ constitutes a number of difficulties for several reasons compared to the investigation of completeness and minimality of this system. There are few results in this direction compared to completeness and minimality results. We note some of these results: using results of a general character for basicity of root vectors of a quadratic operator bundles, A.A.Shkalikov proved that if

$$\alpha \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

then S_α^+ is a Riesz bases in $L_2(0,\pi)$ space. It is proved by L.V.Kritskov that if $\operatorname{Im}\alpha \neq 0$, then S_α^+ is not uniformly miniman and hence a Schauder bases in $L_2(0,\pi)$ space. The most comprehensive result in the study of the Shauder basicity of the Kostyuchenko system is due to B. Bilalov; B.Bilalov obtained a criterion for the basicity of the system S_α^+ in the $L_2(0,\pi)$ space under the natural conditions on the parameter contained in the system.

It should be noted that the system S_α^+ is also of interest from the point of view of its applications in practical tasks, too. For example, the study of the basis properties of the Kostyuchenko system is interesting from the point of view of the theory of optimal control. Similar problems are also encountered in the context of damped oscillations of large mechanical systems. Problems of this kind can be found, for example, in the works of A.G. Butkovsky, L.A.Muravei.

To our knowledge, the concept of frame was introduced by R.J. Duffin and A.C. Schaeffer in connection with the study of some problems of nonharmonic Fourier series related to perturbed exponential systems. In this paper, some properties of frames from exponential systems are established. In this work, the concept of an abstract frame in a Hilbert space was also introduced and some properties of exponential frames were transferred to this case.

In the 80s of the last century applications of wavelets in various fields of natural sciences were found. Wavelets are widely used in the processing and coding of signals, images of various nature (speech, satellite images, X-ray images of internal organs, etc.), pattern recognition, in the study of the properties of surfaces of crystals and nanoobjects, and in many other areas. In connection with numerous applications in various fields of science and technology, the theory of frames is rapidly developing and is of growing interest. As an example of scientific publications devoted to the theory of frames, we cite the monographs of Ch. Chui, Y. Meyer, I. Daubechies, S. Mallat, R. Young, Ch. Heil, O.Christensen et al. and review papers by P. Cazassa et al.

The study of frame properties of families of elements obtained by operator iterations is one of the central problems in "dynamic sampling", which is a relatively new research topic in applied harmonic analysis. This new research topic has attracted considerable attention in recent years (see, for example, papers by the authors Aldroubi, A., Christensen, O., Cabrelli, C., Cakmak, A.F., Molter, U., Hasannasab, M., Philipp, F., Stoeva, D. etc.).

It is known that the concepts of orthogonal and orthonormal systems play an important role in the study of a number of problems related to Hilbert spaces. This applies to the study of the properties of these systems themselves as well as the identification of the properties of various objects through these systems. One of the problems of this type is, for example, the well-known Ringrose characterization of compact operators acting in a Hilbert space in terms of orthonormal systems. This characterization of compact operators states that a bounded linear operator acting on a Hilbert space H is compact if and only if it satisfies

$$\|Ae_n\| \to 0$$

for every orthonormal sequence $\{e_n\}$ in H. Researches in this direction include papers by the authors J.R. Ringrose, P.A. Fillmore and J.P. Williams, J.H. Anderson and J.G. Stampfli, K. Muroi and K. Tamaki, D. Bakic and B. Guljas.

It is well known that every separable Hilbert space possesses an orthonormal Schauder bases, i.e. a Schauder basis $\{x_n\}_{n=1}^{\infty}$ for which

$$\|x_n\| = 1$$

and

$$(x_n, x_m) = 0$$

for every $n,m \in N, n \neq m.$ Besides it, it can easily be shown that every sequence $\{x_n\}_{n=1}^{\infty}$ of elements in any Hilbert space with the properties $\|x_n\| = 1$ and $(x_n, x_m) = 0$ for every $n,m \in N, n \neq m$, is a basic sequence in this space. It is easy to see that the number 1 in this formulation can easily be replaced by the any other positive number by retaining the mentioned property. Therefore, the question of the existence of a basis, where the angles between any two elements are the same, but are not equal to zero, i.e. 0 is replaced by another number arises naturally. As far as we know, the first result in this direction is due to T.E. Khmyleva and I.P.Bukhtina. M.A.Sadybekov and A.M. Sarsenby, studying the question of unconditional basicity, obtained a similar result for almost normalized sequences. Despite the existence of various abstract basicity criteria, checking whether a given sequence is basic or not sometimes presents considerable difficulty for specific sequences of elements. Therefore, obtaining easily verifiable conditions is a very topical problem, and the researches of these authors are interesting from this point of view as well.

The dissertation is devoted to the above range of questions of the theory of basis and frames. Therefore, we consider the topic of the dissertation relevant and of scientific interest.

Object and subject of research. Lebesgue spaces, space of continuous functions, weighted exponential and weighted trigonometric systems, orthonormal and orthonormal type systems, Kostyuchenko-type systems, system of powers, iterations of the multiplication operator.

The goal and objectives of the study. Characterization of weights in connection with the basis properties of exponential and trigonometric systems in Lebesgue spaces, obtaining necessary conditions for the basis property of Kostyuchenko type systems in Lebesgue spaces, basis properties and expansion properties of orthonormal type systems, characterization of compact operators in terms of orthonormal systems, investigation of frame properties of iterations of the multiplication operator.

General technique of studies. When obtaining the main results, the methods of functional analysis, the theory of bases, the theory of frames, the theory of functions of a complex variable, the theory of approximation and harmonic analysis are used.

Main provisions of dissertation.

1. Characterization of weights relative to which the exponential and trigonometric systems are complete and minimal in Lebesgue spaces;

2. Characterization of weights with respect to which the exponential system and the trigonometric system have excesses;

3. Obtainment of necessary conditions for the basicity of Kostyuchenko type systems

$$\left\{\varphi^{n}(t)\sin nt\right\}_{n\in N},$$

$$\left\{\varphi^{n}(t)\cos nt\right\}_{n\in N\cup\{0\}}$$

in $L_p \ (1\le p<+\infty)$ spaces;

4. Investigation of the basicity of systems of the form

$$\{\varphi^{n}(t)\}_{n=0}^{\infty}$$

in the spaces $L_p[a,b]$ and $C[a,b]$;

5. Investigation of the pseudo-basicity of systems of the form

$$\{\varphi^{n}(t)\}_{n=0}^{\infty}$$

in $C[a,b]$;

6. Investigation of the frame property of the orbits

$$\left\{T_{\varphi}^{n} f\right\}_{n=0}^{\infty}$$

and

$$\left\{T_{\varphi}^{n} f\right\}_{n=-\infty}^{\infty}$$

of the multiplication operator

$$T_{\varphi} f(t)=\varphi(t) f(t),\ f\in L_2(a,b)$$

in $L_2(a,b)$ space;

7. Some questions related to the basis properties of pseudo-orthogonal systems in a Hilbert space.

8. Characterization of compact operators in terms of the orthonormal systems.

Scientific novelty.

1. When two of the powers α_j do not satisfy the Makengaupt condition, completeness and minimality in L_p spaces of systems of the form

$$\left\{ \prod_{j=1}^{r} \left| t - t_j \right|^{\alpha_j} e^{\mathrm{int}} \right\}_{n \in Z}$$

are completely investigated;

2. A class of all functions $\omega(t) \in L_p(-\pi,\pi)$ $\left(\omega(t) \in L_p(0,\pi)\right)$, that ensures the completeness and minimality in the corresponding space L_p of the system obtained from the system

$$\{\omega(t)e^{int}\}_Z \ \left(\{\omega(t)\cos nt\}_{n=0}^{\infty}\right)$$

by removing an element, is determined;

3. It is shown that when the system $\{\varphi_n(t)\}$ is a classical system of sines, under the natural condition

$$mes\{t : \omega(t) = 0\} = 0,$$

the fact that $\omega(t)\varphi_n(t) \in L_p$ for any n implies the completeness of $\{\omega(t)\varphi_n(t)\}$ in the space L_p. In addition, it is proved that this fact, whose analogues are well known for classical exponential and cosine systems, are, in general, not true for arbitrary complete or arbitrary complete orthonormal systems;

4. An example of a class of weight functions $\omega(t) \in L_p(0,\pi)$ is given for which the system

$$\{\omega(t)\cos nt\}_{n \in Z_+}$$

is complete in $L_p(0,\pi)$, but neither this system, nor a system obtained by eliminating any finite number of its elements, is complete and minimal in $L_p(0,\pi)$;

5. Necessary condition for the basicity of Kostyuchenko type systems

$$\left\{\varphi^{n}(t)\sin nt\right\}_{n\in N},$$

$$\left\{\varphi^{n}(t)\cos nt\right\}_{n\in N\cup\{0\}},$$

in L_p $(1\le p<+\infty)$ spaces is found; this condition implies, in particular, the necessary condition for the Kostyuchenko system

$$\{e^{i\alpha nt}\sin nt\}_{n\in \mathrm{N}}$$

to be a basis in L_p $(1\le p<+\infty)$ space;

6. It is shown that a system of the form

$$\{\varphi^{n}(t)\}_{n=0}^{\infty}$$

is not a basis in the space $L_p[a,b]$, where $\varphi(t)$ is any measurable almost everywhere finite function on $[a,b]$;

7. It is shown that a system of the form

$$\{\varphi^{n}(t)\}_{n=0}^{\infty}$$

is not a basis in the space $C[a,b]$, where $\varphi(t)$ is any (real or complex-valued) continuous function on $[a,b]$;

8. It is shown that a system of the form

$$\{\varphi^{n}(t)\}_{n=0}^{\infty}$$

cannot be a pseudo-basis in $C[a,b]$, where $\varphi(t)$ is any (real or complex-valued) continuous function on $[a,b]$;

9. The frame properties of iterations

$$\left\{T_{\varphi}^{n} f\right\}_{n=0}^{\infty}$$

and

$$\left\{T_{\varphi}^{n} f\right\}_{n=-\infty}^{\infty}$$

of the multiplication operator

$$T_\varphi f(t) = \varphi(t) f(t), \quad f \in L_2(a,b)$$

in the space $L_2(a,b)$ are studied. It is shown that the orbit

$$\left\{T_\varphi^n f\right\}_{n=0}^{\infty}$$

of the multiplication operator T_φ cannot form a frame for the space $L_2(a,b)$ for any measurable generator $\varphi(t)$ and any $f \in L_2(a,b)$. All frames of the form

$$\{\varphi^n(t)\}_{n=-\infty}^{\infty}$$

that are iterations of the multiplication operator T_φ are characterized. It is shown that this problem can be reduced to the following one: *Find (or describe a class) of all real functions* $\alpha(t)$, *for which*

$$\left\{e^{in\alpha(t)}\right\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$. A partial answer to this problem is given.

10. Basicity properties of sequences of elements in a Hilbert space for which the angles between any two distinct elements are equal to the same nonzero number, and the basicity properties of sequences $\{x_n\}_{n=1}^{\infty}$ that have a bounded subsequence $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ such that

$$\left|\left(x_{n_k}, x_{n_m}\right)\right| = a > 0$$

for all sufficiently large $k, m \in N, k \neq m$ are studied;

11. A brief, simpler, and elementary proof of a characterization of compact operators in terms of orthonormal sequences is given; the proof of this fact presented in the dissertation shows that this fact is also true for operators acting from a Hilbert space into some Banach (which may not be Hilbert) space.

Theoretical and practical value of the study. The dissertation is of a theoretical character. Its results can be used in approximation theory, in frame theory, in the spectral theory of differential operators, to justify the Fourier method in solving differential equations, etc.

Approbation and application. The main results of the dissertation were reported at scientific seminars of the department "Non-harmonic analysis" of Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan (Doctor of Physical and Mathematical Sciences, Corresponding Member of Azerbaijan National Academy of Sciences, Professor B.T. Bilalov), at seminars of the department "Functional Analysis" (Doctor of Physical and Mathematical Sciences, Professor N.Sh.Iskenderov), at the weekly seminars of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan and at various international scientific conferences.

Personal contribution of the author. All results obtained in the dissertation are the personal contribution of the author.

Publications of the author. The main results of the dissertation were published in 27 papers .

The name of the institution where the dissertation was completed. The work was carried out in the department "Non-harmonic analysis" of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Volume and structure of the dissertation (in signs, indicating the volume of each structural unit separately).

The total volume of dissertation work is 378927 characters (title page - 433 characters, content 2494 characters, introduction - 66000 characters, first chapter - 68000 characters, second chapter - 122000 characters, third chapter - 54000 characters, fourth chapter - 38000 characters, fifth chapter - 26000 characters, conclusions - 2000 characters). The list of used literature consists of 139 items.

THE CONTENT OF THE DISSERTATION

The thesis consists of an introduction, five chapters and a list of literature.

The introduction demonstrates the relevance of the topic, provides an overview of works related to the topic of the dissertation, and outlines the summary of the work.

Before proceeding to the description of the paragraphs, we recall some definitions:

Definition 1. *A sequence of vectors* $\{x_n\}_{n=1}^{\infty}$ *in a Banach space* X *is called a Schauder basis for* X *if for each vector* x *there is a unique sequence of scalars* $\{\alpha_n\}_{n=1}^{\infty}$ *such that*

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n + \ldots$$

in the norm of the space X *.*

Definition 2. *A sequence of vectors* $\{x_n\}_{n=1}^{\infty}$ *in a Banach space* X *is called a basic sequence if it is a Schauder basis for the closure of its linear span.*

Definition 3. *Let* $\{x_n\}_{n\in N}$ *be a system of elements in a Banach space* B*. Denote by* B_k *the closure of the linear span of* $\{x_n\}_{n\in N, n\neq k}$*. If there exists* $\delta > 0$ *such that*

$$\inf_{y\in B_k} \|x_k - y\| \geq \delta \|x_k\|$$

for all $k \in N$*, then* $\{x_n\}_{n\in N}$ *is called a uniformly minimal system in* B *.*

Definition 4. *Let* $\{x_n^+, x_n^-\}_{n\geq 0}$ *be a "double" system of elements in a Banach space* B*. A system* $\{x_n^+, x_n^-\}_{n\geq 0}$ *is called a basis for* B *if for each* $x \in B$ *there is a unique sequence of complex numbers* $\{a_n^+, a_n^-\}_{n\geq 0}$ *such that*

$$\left\| \sum_{n=0}^{N^+} a_n^+ x_n^+ + \sum_{n=0}^{N^-} a_n^- x_n^- - x \right\| \to 0 \text{ as } N^+, N^- \to \infty.$$

Definition 5. *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in a Banach space* X *with* $x_n \neq 0$ *is called a pseudo-basis (or representation system) for* X *if for each vector* $x \in X$ *there is a sequence of scalars* $\{\alpha_n\}_{n=0}^{\infty}$ *such that*

$$x = \alpha_0 x_0 + ... + \alpha_n x_n + ...$$

Obviously, every Schauder basis is a pseudobasis, but the converse is not true in general. In particular, it is known that every separable Banach space contains a pseudobasis; but it is well known that not every separable Banach space has a Schauder basis.

Definition 6. *A sequence* $\{x_n\}_{n=0}^{\infty}$ *in a Hilbert space* H *is called a frame for* H *if there are constants* $A, B > 0$ *such that*

$$A\|x\|^2 \leq \sum_{n=0}^{\infty} |(x, x_n)|^2 \leq B\|x\|^2$$

for all $x \in H$ *; if* $A = B$*, then the frame* $\{x_n\}_{n=0}^{\infty}$ *is called* A *- tight frame.*

Chapter I is dedicated to the study of basicity properties (completeness, minimality, Schauder basicity) of weighted exponential systems of a concrete form in Lebesgue spaces.

This chapter consists of three paragraphs.

Section 1.1 is dedicated to the investigation of completeness and minimality of system of the form

$$\left\{ \prod_{j=1}^{r} \left| t - t_j \right|^{\alpha_j} e^{\text{int}} \right\}_{n \in Z}.$$

In this paragraph, we consider a system of the form

$$\left\{ \omega(t) e^{\text{int}} \right\}_{n \in Z}, \tag{1}$$

where

$$\omega(t)=|t-t_1|^{\alpha_1}\,|t-t_2|^{\alpha_2}\,\prod_{j=3}^{r}|t-t_j|^{\alpha_j}\,e^{\mathrm{int}},$$

$$t_j\in[-\pi,\pi]$$

for all $1\leq j\leq r,$

$$\alpha_1,\alpha_2\in\left[\frac{1}{q},1+\frac{1}{q}\right),$$

and

$$\alpha_j\in\left(-\frac{1}{p},\frac{1}{q}\right)$$

for all $3\leq j\leq r$.

Note that, in the sequel, we denote by Q the set of all rational numbers.

Theorem 1. *Consider the system*

$$\left\{|t-t_1|^{\alpha_1}|t-t_2|^{\alpha_2}\,e^{\mathrm{int}}\right\}_{n\in Z},$$

where

$$t_1,t_2\in[-\pi,\pi]$$

and

$$\alpha_1,\alpha_2\in\left[\frac{1}{q},1+\frac{1}{q}\right).$$

The following statements hold:

1) *If* $\frac{t_2-t_1}{\pi}\notin Q$, *then the system*

$$\left\{|t-t_1|^{\alpha_1}\,|t-t_2|^{\alpha_2}\,e^{\mathrm{int}}\right\}_{n\in Z/\{k_1;k_2\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *for any choice of indices* k_1 *and* k_2;

2) *If* $|t_2 - t_1| = 2\pi$ *, then the system*

$$\left\{|t-t_1|^{\alpha_1}|t-t_2|^{\alpha_2} e^{\mathrm{int}}\right\}_{n\in Z/\{k_0\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *for any integer* k_0*;*

3) *If* $t_2 - t_1 = 2\pi\frac{k}{m}$*, where* $m \neq 1$ *and* $(k,m)=1$*, then*

$$\left\{|t-t_1|^{\alpha_1}|t-t_2|^{\alpha_2} e^{\mathrm{int}}\right\}_{n\in Z/\{k_1;k_2\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *if and only if* $k_2 \neq k_1 (\mathrm{mod}\, m)$

.

The proof of this theorem, presented in the dissertation, shows that the following more general statement is also true:

Theorem 2. *Consider the system* (1). *The following statements hold:*

1) *If* $\frac{t_2 - t_1}{\pi} \notin Q$*, then the system*

$$\left\{\omega(t)\cdot e^{\mathrm{int}}\right\}_{n\in Z/\{k_1;k_2\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *for any choice of indices* k_1 *and* k_2*;*

2) If $|t_2 - t_1| = 2\pi$ *, then the system*

$$\left\{\omega(t)\cdot e^{\mathrm{int}}\right\}_{n\in Z/\{k_0\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *for any integer* k_0*;*

3) If $t_2 - t_1 = 2\pi\frac{k}{m}$*, where* $m \neq 1$ *and* $(k,m)=1$*, then*

$$\left\{\omega(t)\cdot e^{\mathrm{int}}\right\}_{n\in Z/\{k_1;k_2\}}$$

is complete and minimal in $L_p(-\pi,\pi)$ *if and only if* $k_2 \neq k_1 (\mathrm{mod}\, m)$.

Section 1.2 is devoted to the determination of the class of all functions $\omega(t) \in L_p(-\pi, \pi)$ for which the system

$$\left\{\omega(t)e^{\mathrm{int}}\right\}_{n\in Z}$$

becomes complete and minimal in the space $L_p(-\pi, \pi)$, $1 < p < \infty$, when exactly one of its elements is eliminated.

The purpose of this section is to prove the following fact.

Theorem 3. *Let* k_0 *be any integer. The system*

$$\left\{\omega(t)e^{\mathrm{int}}\right\}_{n\in Z/\{k_0\}}$$

is complete and minimal in the space $L_p(-\pi, \pi)$, $1 < p < \infty$, *if and only if*

$$\omega(t) \in L_p(-\pi, \pi),$$

$$\frac{1}{\omega(t)} \notin L_q(-\pi, \pi)$$

and in addition,

1) *there is a (unique) point* $t_0 \in [-\pi, \pi]$ *such that*

$$\frac{t - t_0}{\omega(t)} \in L_q(-\pi, \pi);$$

or

2) $\dfrac{(t-\pi)(t+\pi)}{\omega(t)} \in L_q(-\pi, \pi).$

The following corollary immediately follows from this theorem.

Corollary 1. *If a system*

$$\left\{\omega(t)e^{\mathrm{int}}\right\}_{n\in Z}$$

becomes complete and minimal in $L_p(-\pi,\pi)$ *when one of its elements is eliminated, then it also becomes complete and minimal in* $L_p(-\pi,\pi)$ *when any one of its elements is eliminated.*

Section 1.3 is devoted to the study of the Schauder basicity in Lebesgue spaces of systems of the form

$$\left\{\omega(t)e^{\mathrm{int}}\right\}_{n\in Z}$$

after elimination of a finite number of its elements.

The purpose of this section is to prove the following proposition.

Theorem 4. *Let* $\omega(t)$ *be any function and* $n_1,...,n_k$ *be any integers. Then the system*

$$\{\omega(t)e^{int}\}_{n\in Z/\{n_1,\ldots,n_k\}}$$

is not a Schauder basis in the space $L_p(-\pi,\pi), 1< p<\infty$.

It should be noted that, despite the independence of the proofs given in this dissertation, the assertion of this theorem and the theorem of section 2.3 follow from a more general similar result of the papers published earlier by K.S. Kazarian, and this result is not indicated in the provisions that are presented for the defense of this dissertation and are given here only to clarify the question of the basicity of a degenerate exponential and trigonometric systems with excess, which naturally arises after discussing the completeness and minimality of such systems.

Chapter II is dedicated to the study of basis properties (completeness, minimality, and Schauder basis property) of weighted trigonometric (sine, cosine) systems with a general weight.

This chapter consists of four paragraphs.

Section 2.1 is dedicated to the study of the completeness of a degenerate system of sines.

Theorem 5. *Let* $\omega(t)$ *be any measurable function on* $(0,\pi)$ *such that*

1) $mes\{t:\omega(t)=0\}=0$

and

2) $\omega(t) sinnt \in L_p(0,\pi), \forall n \in N$.

Then a system

$$\{\omega(t)\sin nt\}_{n\in N}$$

is complete in $L_p(0,\pi)$.

It should be noted that this statement and well-known similar results for the exponential system and the cosine system show that the system $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ are exponential or trigonometric (cosine or sine) systems, becomes complete in the corresponding Lebesgue space $L_p(-\pi,\pi)$ or $L_p(0,\pi)$, respectively, whenever

$$mes\{t:\omega(t)=0\}=0$$

and $\omega(t)\varphi_n(t)$ belong to the corresponding Lebesgue space for all indices n. This might give the impression that if the function $\omega(t)$ is such that

$$mes\{t:\omega(t)=0\}=0$$

and

$$\omega(t)\varphi_n(t)\in L_p(a,b)$$

for all indices n, where $\{\varphi_n(t)\}$ is any complete system in $L_p(a,b)$, then the system $\{\omega(t)\varphi_n(t)\}$ is also complete in the space $L_p(a,b)$. The results obtained in this dissertation show that this is not the case in general.

Theorem 6. *There is a complete system* $\{\varphi_n(t)\}$ *in the space* $L_p(-\pi,\pi)$ *and a measurable function* $\omega(t)$ *such that*

$$mes\{t:\omega(t)=0\}=0$$

and

$$\omega(t)\varphi_n(t)\in L_p(-\pi,\pi)$$

for all indices n, *but the system* $\{\omega(t)\varphi_n(t)\}$ *is not complete in* $L_p(-\pi,\pi)$.

Using the proof of this theorem and applying the Schmidt orthogonalization process to the system given in the proof of this theorem, we obtain the following fact.

Theorem 7. *There is a complete orthonormal system* $\{\varphi_n(t)\}$ *in the space* $L_2(-\pi,\pi)$ *and a measurable function* $\omega(t)$ *such that*

$$mes\{t:\omega(t)=0\}=0$$

and

$$\omega(t)\varphi_n(t)\in L_2(-\pi,\pi)$$

for all indices n, *but the system* $\{\omega(t)\varphi_n(t)\}$ *is not complete in* $L_2(-\pi,\pi)$.

In addition to these results, we obtain the following descriptions of the largest set of functions $\omega(t)$ for which the system

$$\{\omega(t)\sin nt\}_{n\in N}$$

is a complete system in $L_p(0,\pi)$:

Theorem 8. *The system*

$$\{\omega(t)\sin nt\}_{n\in N}$$

is complete in $L_p(0,\pi)$ *if and only if*

$$t(t-\pi)\omega(t)\in L_p(0,\pi)$$

and

$$mes\{t:\omega(t)=0\}=0.$$

Theorem 9. *The system*

$$\{\omega(t)\sin nt\}_{n\in N}$$

is complete in $L_p(0,\pi)$ *if and only if*

$$\omega(t)\sin t\in L_p(0,\pi)$$

and

$$mes\{t:\omega(t)=0\}=0.$$

The main goal of Section 2.2 is to determine the class of all functions $\omega(t)$ for which the system

$$\{\omega(t)\cos nt\}_{n=0}^{\infty}$$

becomes complete and minimal in the space $L_p(0,\pi)$, $1<p<\infty$, when one of its elements is eliminated.

For simplicity, we denote

$$R_k = \{t : t \in [0,\pi], \cos kt = 0\}.$$

The main goal of this section is to prove the following theorem.

Theorem 10. *Let* $k_0 \in Z_+$ *be an arbitrary number. A system*

$$\{\omega(t)\cos nt\}_{n\in Z_+/\{k_0\}}$$

is complete and minimal in $L_p(0,\pi)$, $1<p<\infty$, *if and only if*

$$\omega(t) \in L_p(0,\pi),$$

$$\frac{1}{\omega(t)} \notin L_q(0,\pi)$$

and, in addition

1) $\dfrac{t^2}{\omega(t)} \in L_q(0,\pi)$;

or

2) $\dfrac{(t-\pi)^2}{\omega(t)} \in L_q(0,\pi)$;

or

3) *there is a (unique) point* $t_0 \in (0,\pi)$ *such that* $t_0 \notin R_{k_0}$ *and*

$$\frac{t-t_0}{\omega(t)} \in L_q(0,\pi).$$

Proposition 1. *Let the function* $\omega(t) \in L_p(0,\pi)$ *and the point* $t_0 \in (0,\pi)$ *be such that*

$$\frac{1}{\omega(t)} \notin L_q(0,\pi)$$

and

$$\frac{t-t_0}{\omega(t)} \in L_q(0,\pi).$$

If $t_0 \in R_{k_0}$, *then the system*

$$\{\omega(t)\cos nt\}_{n\in Z_+ /\{k_0\}}$$

is neither complete nor minimal in $L_p(0,\pi)$.

As a consequence of these statements, we obtain the validity of the following corollary, which describes the set of all functions $\omega(t)$ for which the system

$$\{\omega(t)\cos nt\}_{n\in Z_+}$$

becomes complete and minimal in $L_p(0,\pi)$ when any of its elements is eliminated.

Corollary 2. *A system*

$$\{\omega(t)\cos nt\}_{n\in Z_+}$$

is complete and minimal in $L_p(0,\pi)$, $1<p<\infty$, *when any of the elements is eliminated if and only if*

$$\frac{1}{\omega(t)} \notin L_q(0,\pi)$$

and in addition

1) $\dfrac{t^2}{\omega(t)} \in L_q(0,\pi)$

or

2) $\dfrac{(t-\pi)^2}{\omega(t)} \in L_q(0,\pi)$

or

3) *there is a point* $t_0 \in (0,\pi)$ *such that*

$$t_0 \notin \bigcup_{k=1}^{\infty} R_k$$

and

$$\frac{t-t_0}{\omega(t)} \in L_q(0,\pi).$$

Section 2.3 is dedicated to the Schauder basicity of degenerate trigonometric systems with excess.

To prove the main result of the first part of this section, we use some auxiliary facts. One of these facts is the following

Lemma 1. *Let* $\omega(t)$ *be any non-trivial measurable function in* $L_p(0,\pi)$, $1 \le p < \infty$. *Then*

$$\inf_{n \in Z_+} \|\omega(t)\cos nt\|_{L_p} \neq 0.$$

Theorem 11. *Let* $\omega(t)$ *be any function and* $k_0 \in Z_+$ *be any integer. Then the system*

$$\{\omega(t)\cos nt\}_{n \in Z_+ / \{k_0\}}$$

is not a Schauder basis in $L_p(0,\pi)$, $1 < p < \infty$.

The second part of Section 2.3 is dedicated to the study of the Schauder basicity of degenerate systems of sines.

To prove the main result of this part of this section, we use the following auxiliary fact.

Lemma 2. *Let* $\omega(t)$ *be any non-trivial measurable function on* $[0,\pi]$ *such that*

$$\omega(t)\sin nt \in L_p(0,\pi), 1 < p < \infty,$$

for all $n \in N$. *Then*

$$\inf_{n\in N}\left\|\omega(t)\sin nt\right\|_{L_p(0,\pi)} \neq 0.$$

The main assertion of this part is the following fact.

Theorem 12. *Let $\omega(t)$ be any measurable function and k_0 be any natural number. Then the system*

$$\{\omega(t)\sin nt\}_{n\in N/\{k_0\}}$$

is not a Schauder basis in $L_p(0,\pi)$.

Section 2.4 provides an example of a weight function $\omega(t)\in L_p(0,\pi)$ for which the system

$$\{\omega(t)\cos nt\}_{n\in Z_+}$$

is complete in $L_p(0,\pi)$, but neither this system, nor the system obtained by eliminating any finite number of its elements, is complete and at the same time minimal in $L_p(0,\pi)$.

The main result of this section is the following fact.

Theorem 13. *Let $\omega(t)$ be any continuous function defined on $[0,\pi]$ that is infinitely differentiable at zero, $\omega(t)\neq 0$ almost everywhere, and*

$$\omega^{(n)}(0)=0,\ \forall n\in Z_{+.}$$

Then the system

$$\{\omega(t)\cos nt\}_{n\in Z_+}$$

is complete, but not minimal, and cannot be made complete and minimal in $L_p(0,\pi)$ by eliminating a finite number of its elements.

It is known that the set of functions $\omega(t)$ that are continuous on $[0,\pi]$, are infinitely differentiable at zero, $\omega(t)\neq 0$ almost everywhere, and

$$\omega^{(n)}(0)=0,\ \forall n\in Z_{+.}$$

is not empty. For example, the following function

$$\omega(t)=\begin{cases} e^{-\frac{1}{t^2}}, & if \quad t\neq 0; \\ 0, & if \quad t=0. \end{cases}$$

satisfies all of these conditions.

Chapter III is dedicated to the study of the basicity properties (completeness, minimality, basis property, etc.) of systems of the form

$$\left\{\varphi^n(t)\sin nt\right\}_{n\in N},$$
$$\left\{\varphi^n(t)\cos nt\right\}_{n\in N\cup\{0\}},$$

where $\varphi:[a,b]\to C$ is a measurable, almost everywhere finite function and a,b are some real numbers, and of a double system of functions of the form

$$\left\{A(t)\varphi^n(t);B(t)\overline{\varphi}^n(t)\right\}_{n\geq 0}$$

in Lebesgue spaces $L_p=L_p(a,b)$, $1\leq p<\infty$, and basicity properties of systems of the form

$$\left\{\varphi^n(t)\right\}_{n\in N}$$

in Lebesgue spaces and spaces of continuous functions.

This chapter consists of five paragraphs.

Section 3.1 is dedicated to the study of necessary conditions for the basicity of power systems of the form

$$\left\{A(t)\varphi^n(t);B(t)\overline{\varphi}^n(t)\right\}_{n\geq 0}$$

in Lebesgue spaces.

Consider the following system of functions

$$\left\{A(t)\varphi^n(t);B(t)\overline{\varphi}^n(t)\right\}_{n\geq 0}. \tag{2}$$

Suppose that the complex functions $A(t)$, $B(t)$, and $\varphi(t)$ satisfy the following conditions:

1) The functions $|A(t)|$, $|B(t)|$ are measurable on (a,b); besides this

$$\sup vrai\left\{|A(t)|^{\pm 1}, |B(t)|^{\pm 1}\right\} < \infty;$$

2) $\varphi(t)$ is a continuous function on $[a,b]$.

First, consider the basis property of systems of the form (2) in L_p $(1 \le p < \infty)$. The main result in this direction is as follows:

Theorem 14. *Assume that* $A(t)$, $B(t)$ *and* $\varphi(t)$ *satisfy conditions* 1) *and* 2). *If the system* (2) *is a basis in* L_p, *then* $|\varphi(t)| \equiv const$ *on* $[a,b]$.

Let the symbols I or J denote an empty or finite set. We put

$$A_1(t) = A(t) \prod_{i \in I} |t - \xi_i|^{\alpha_i}, \; t \in [a,b],$$

$$B_1(t) = B(t) \prod_{j \in J} |t - \theta_j|^{\beta_j}, \; t \in [a,b],$$

where $A(t)$, $B(t)$ satisfy condition 1), $\xi_i (i \in I)$, $\theta_j (j \in J)$ are some points in $[a,b]$, and $\alpha_i (i \in I)$, $\beta_j (j \in J)$ are some scalars.

The following generalization of Theorem 14 is valid.

Theorem 15. *If the system*

$$\left\{A_1(t)\varphi^n(t); B_1(t)\overline{\varphi}^n(t)\right\}_{n \ge 0}$$

is a basis in L_p, *then* $|\varphi(t)| \equiv const$ *on* $[a,b]$.

These results allow one to obtain similar results for weighted spaces $L_{p,\rho(t)}(a,b)$, where $1 \le p < \infty$, and $\rho(t)$ is a non-negative continuous functions with a finite number of degenerations, more precisely

$$\rho(t) \equiv \prod_{i \in I} |t - \mu_i|^{\omega_i},$$

where μ_i are some points in $[a,b]$ and ω_i are some scalars.

Theorem 16. *If the system* (2) *is a basis in* $L_{p,\rho(t)}(a,b)$, *then* $|\varphi(t)| \equiv const$ *on* $[a,b]$.

It should be noted that the approach used in Section 3.3 shows that the results of Section 3.1 remain valid under more general assumptions. Namely, in Section 3.1 the function $\varphi(t)$ is assumed to be continuous; but in fact all the results of this section are true under a weaker restriction on $\varphi(t)$, i.e., if $\varphi(t)$ is simply a measurable and almost everywhere finite function.

Section 3.2 is dedicated to the study of necessary conditions for basicity of the Kostyuchenko system

$$\left\{e^{i\alpha nt}\sin nt\right\}_{n\in N}$$

and systems of a more general form

$$\left\{\varphi^{n}(t)\sin nt\right\}_{n\in N}$$

in Lebesgue spaces $L_p = L_p(a,b)$, $1 \le p < \infty$, where $\varphi:[a,b]\to C$ is a measurable, almost everywhere finite function, and a, b are some real numbers.

One of the results of this section is the following theorem.

Theorem 17. *If the system*

$$\left\{\varphi^{n}(t)\sin nt\right\}_{n\in N}$$

is a basis in L_p, *then* $|\varphi(t)| \equiv const$ *almost everywhere on* $[a,b]$.

To prove this theorem, we use the following lemma.

Lemma 3. *Let* $E \subset [a,b]$ *be a Lebesgue measurable subset of* $[a,b]$. *If there exists a subsequence* $\{n_k\}$ *of natural numbers and a number* p $(1 \le p < \infty)$ *such that*

$$\int_E |\sin n_k t|^p\, dt \to 0 \text{ as } k \to \infty,$$

then $mesE = 0$.

Note that a similar lemma is true for the cosine system.

Lemma 4. *Let* $E \subset [a,b]$ *be a Lebesgue measurable subset of* $[a,b]$. *If there exists a subsequence* $\{n_k\}$ *of natural numbers and a number* p $(1 \le p < \infty)$ *such that*

$$\int_E |\cos n_k t|^p \, dt \to 0 \text{ as } k \to \infty,$$

then $mesE = 0$.

Using Lemma 4 instead of Lemma 3, it is easy to see that a similar statement is also true for system

$$\{\varphi^n(t)\cos nt\}_{n \in N \cup \{0\}}.$$

Theorem 18. *If the system*

$$\{\varphi^n(t)\cos nt\}_{n \in N \cup \{0\}}$$

is a basis in L_p, *then* $|\varphi(t)| \equiv const$ *almost everywhere on* $[a,b]$.

From these results we obtain, in particular, the following result for the systems S_α^+ and

$$C_\alpha^+ = \{e^{i\alpha nt}\cos nt\}_{n \in N \cup \{0\}},$$

which covers cases of all spaces L_p.

Corollary 3. *If* $\operatorname{Im}\alpha \ne 0$, *then systems* S_α^+ *and* C_α^+ *are not bases in the space* L_p.

Section 3.3 is dedicated to the study of the basis property of the system of powers

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

in the Lebesgue spaces L_p.

The main statement of this section is as follows.

Theorem 19. *Let* $\varphi(t)$ *be any measurable almost everywhere finite function on* $[a,b]$. *Then*

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

is not a basis in the space L_p.

It should be noted that the approach used in this section shows that the results of Section 3.1 remain valid under more general assumptions. Namely, in Section 3.1 the function $\varphi(t)$ must be a continuous function; but in fact all the results of this section are true under a weaker assumption - provided that $\varphi(t)$ is simply a measurable and almost everywhere finite function.

Section 3.4 is dedicated to the study of the Schauder basicity of systems of the form

$$\left\{\varphi^n(t)\right\}_{n=0}^{\infty}$$

in the space of continuous functions $C[a,b]$.

The exact formulation of the main statement is as follows:

Theorem 20. *Let* $\varphi(t)$ *be any continuous function on the segment* $[a,b]$. *Then the system*

$$\left\{\varphi^n(t)\right\}_{n=0}^{\infty}$$

is not a basis in the space of continuous functions $C[a,b]$.

The purpose of Section 3.5 is to prove that in the most general case of any continuous function $\varphi(t)$ defined on a segment $[a,b]$, the system of powers

$$\left\{\varphi^n(t)\right\}_{n=0}^{\infty}$$

cannot even be a pseudo-basis in $C[a,b]$.

Theorem 21. *Let* $\varphi(t)$ *be any (real or complex) continuous function on* $[a,b]$. *Then*

$$\left\{\varphi^n(t)\right\}_{n=0}^{\infty}$$

is not a pseudo-basis in $C[a,b]$.

The fourth chapter is dedicated to the study of frame properties of iterations of the multiplication operator

$$T_\varphi f(t) = \varphi(t) f(t), \quad f \in L_2(a,b).$$

This chapter consists of two paragraphs.

The purpose of Section 4.1 is to show that the orbit

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

cannot form a frame for the space $L_2(a,b)$ for any measurable generator $\varphi(t)$ and any $f \in L_2(a,b)$.

The following propositions play a crucial role in the proof of the main result of this section.

Lemma 5. *Let φ and f be any measurable functions. If the system*

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

is a pseudo-basis in $L_p(a,b)$, $1 \le p < \infty$, then $|\varphi(t)| \equiv const$ almost everywhere on $[a,b]$.

Lemma 6. *Let φ and f be any measurable functions. If*

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

is a frame in $L_2(a,b)$, then $|\varphi(t)| \equiv 1$ almost everywhere on $[a,b]$.

The main result of this section is the following theorem.

Theorem 22. *Let $\varphi(t)$ be any measurable function and f any square-summable function on $[a,b]$. The system*

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

cannot be a frame in $L_2(a,b)$.

It follows from this theorem, in particular, that a system of the form

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

cannot be a frame in $L_2(a,b)$. The classical exponential system shows that the situation is completely different if we consider systems of the form

$$\{\varphi^n(t)\}_{n=-\infty}^{\infty}$$

instead of

$$\{\varphi^n(t)\}_{n=0}^{\infty}.$$

Section 4.2 is dedicated to the problem of characterization of frames of the form

$$\left\{\varphi^{n}(t)\right\}_{n=-\infty}^{\infty}.$$

One of the main results of Section 4.2 is the following theorem.

Theorem 23. *Let $\varphi(t)$ be a measurable function on $[a,b]$. If*

$$\left\{\varphi^{n}(t)\right\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$, then $|\varphi(t)|=1$ almost everywhere on $[a,b]$, i.e., the function $\varphi(t)$ is an exponential function of the form

$$\varphi(t)=e^{i\alpha(t)},$$

where $\alpha(t)$ is a real function.

This proposition shows that a characterization of frames of the form

$$\left\{\varphi^{n}(t)\right\}_{n=-\infty}^{+\infty}$$

in the space $L_2(a,b)$ is equivalent to determining the class of all real functions $\alpha(t)$ for which the system

$$\left\{e^{in\alpha(t)}\right\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$.

We give a partial answer to this problem by providing a sufficient condition for the function $\alpha(t)$ for which the system

$$\left\{e^{in\alpha(t)}\right\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$.

Theorem 24. *Let a function $\alpha(t)$, defined on $[a,b]$, be an invertible function whose inverse $\xi:[p,q]\to[a,b]$ satisfies the following conditions:*

1) $\xi(t)$ is an absolutely continuous, strictly increasing function on $[p,q]$, $\xi(p)=a$ and $\xi(q)=b$;

2) there are constants $A,\ B>0$ such that $A\le\xi'(t)\le B$ a.e. on $[p,q]$;

3) $\left|p-q\right| \leq 2\pi.$

Then the system

$$\left\{e^{in\alpha(t)}\right\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$.

Example 1. Let $a=0,\ b=2\pi$ and $\alpha(t)=\sqrt{t+1}$. Then one can easily check that all the conditions imposed on the function $\alpha(t)$ are satisfied. Therefore, system

$$\left\{e^{in\sqrt{t+1}}\right\}_{n=-\infty}^{\infty}$$

is a frame in $L_2(0,2\pi)$.

The following example shows the importance of condition 2) in Theorem 24.

Example 2. A system

$$\left\{e^{in\sqrt{t}}\right\}_{n=-\infty}^{\infty},$$

for which condition 2) is not satisfied, is not a frame in $L_2(0,2\pi)$.

The following example shows the importance of condition 3) in the theorem.

Example 3. A system of exponents

$$\left\{e^{in(2t+1)}\right\}_{n=-\infty}^{\infty},$$

for which condition 3) is not satisfied, is not complete and, therefore, is not a frame in $L_2(0,2\pi)$.

Chapter V is dedicated to the investigation of basis properties and representation properties of certain systems of orthonormal type and the characterization of compact operators by means of orthonormal sequences.

This chapter consists of three paragraphs.

Section 5.1 is dedicated to the study of basis properties of a sequence of elements in a Hilbert space for which the angles between any two elements are equal to the same nonzero number.

In this section, we propose some refinements of the formulation of a result of T.E. Khmyleva and I.P. Bukhtina; a shorter and simpler proof is given, which makes it possible to obtain a generalization of this result.

First, we formulate some facts that are also of independent interest.

Proposition 2. *Let H be a Hilbert space, $\{x_n\}_{n=1}^{\infty}$ a sequence of elements of the space H that satisfies the conditions:*

a) $\|x_n\| = 1$ *for any* $n \in N$,

b) $(x_n, x_m) = a$, $n, m \in N, n \neq m$, *where* a *is some number such that* $a \neq 1$.

Then the system of elements $\{x_n\}_{n=1}^{\infty}$ is ω - linearly independent.

Proposition 3. *Let H be a Hilbert space, $\{x_n\}_{n=1}^{\infty}$ a sequence of elements of the space H that satisfies the conditions:*

a) $\|x_n\| = 1$ *for any* $n \in N$,

b) $(x_n, x_m) = a$ *for* $n, m \in N, n \neq m$.

Then the number a is non-negative.

Theorem 25. *Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in H with the following properties:*

1) $\|x_n\| = 1$ *for all* $n \in N$;

2) $(x_n, x_m) = a$ *for all* $n, m \in N, n \neq m$ *and* $a \neq 0$.

Then $\{x_n\}_{n=1}^{\infty}$ is not a basic sequence in H.

The proof of this theorem given in the dissertation shows the validity of the following more general result.

Theorem 26. *Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in H with the following properties:*

1) $\|x_n\| = 1$ *for all* $n \in N$;

2) $(x_n, x_m) = a$ *for all* $n, m \in N, n \neq m$ *and* $a \neq 0.$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *is not a representation system in the subspace of* H *generated by these elements (and therefore also in the space* H *).*

Section 5.2 is dedicated to the study of the basicity in Hilbert spaces of sequences $\{x_n\}_{n=1}^{\infty}$ that have a bounded subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that

$$\left|\left(x_{n_k}, x_{n_m}\right)\right| = a > 0$$

for all sufficiently large $k, m \in N, k \neq m.$

The main result of Section 5.2 is the following theorem.

Theorem 27. *Let* H *be a Hilbert space and let a bounded sequence of its elements* $\{x_n\}_{n=1}^{\infty}$ *satisfy*

$$\left|\left(x_n, x_m\right)\right| = a > 0$$

for all $n, m \in N, n \neq m.$ *Then* $\{x_n\}_{n=1}^{\infty}$ *is not a basic sequence (and hence a Schauder basis) in* $H.$

Theorem 27 shows that the following more general result is also true.

Theorem 28. *Let* H *be a Hilbert space and* $\{x_n\}_{n=1}^{\infty}$ *have a bounded subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *such that*

$$\left|\left(x_{n_k}, x_{n_m}\right)\right| = a > 0$$

for all sufficiently large $k, m \in N, k \neq m.$ *Then* $\{x_n\}_{n=1}^{\infty}$ *is not a basic sequence (and hence a Schauder basis) in* $H.$

It should be noted that there are specific examples of sequences demonstrating that, in contrast to the analogous result of the previous paragraph, a sequence of elements $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space that has a bounded subsequence $\left\{x_{n_k}\right\}_{k=1}^{\infty}$ such that

$$\left|\left(x_{n_k}, x_{n_m}\right)\right| = a > 0$$

for all sufficiently large $k, m \in N, k \neq m$ can be a representation system in the subspace of H generated by these elements (and hence also in the space H).

Section 5.3 is dedicated to the characterization of compact operators by means of orthonormal sequences.

The well-known Ringrose characterization of compact operators on a Hilbert space states that a bounded linear operator A on a Hilbert space H is compact if and only if it satisfies $\|Ae_n\| \to 0$ for every orthonormal sequence $\{e_n\}$ in H.

Note that the proofs of this theorem known to us do not allow the formulation of a similar statement for operators acting from a Hilbert space into some Banach space, which is not necessarily a Hilbert space.

In this section, we demonstrate a short and simple proof of the Ringrose characterization of compact operators. It should be noted that our proof shows that the assertion of this theorem remains valid for operators acting from a Hilbert space to a Banach space (which is not necessarily a Hilbert space).

Theorem 29. *A linear (not necessarily bounded) operator acting from a Hilbert space H into a Banach space B is compact if and only if it satisfies $\|Ae_n\| \to 0$ for every orthonormal sequence $\{e_n\}$ in H.*

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CONCLUSIONS

The main purpose of the dissertation is to characterize the weights associated with the basicity properties of exponential and trigonometric systems in Lebesgue spaces, to obtain necessary conditions for the basicity of Kostyuchenko-type systems in Lebesgue spaces, to study the frame properties of iterations of the multiplication operator, to study the basis and expansion properties of systems of orthonormal type, to characterize compact operators in terms of orthonormal systems.

The following results are obtained in the dissertation work:

1. When two of the powers α_j do not satisfy the Makengaupt condition, completeness and minimality in L_p spaces of systems of the form

$$\left\{ \prod_{j=1}^{r} \left| t - t_j \right|^{\alpha_j} e^{\mathrm{int}} \right\}_{n \in Z}$$

are completely investigated;

2. A class of all functions $\omega(t) \in L_p(-\pi, \pi)$ $\left(\omega(t) \in L_p(0, \pi)\right)$, that ensures the completeness and minimality in the corresponding space L_p of the system obtained from the system $\{\omega(t) e^{int}\}_Z$ $\left(\{\omega(t) \cos nt\}_{n=0}^{\infty}\right)$ by removing an element, is determined;

3. It is shown that when the system $\{\varphi_n(t)\}$ is a classical system of sines, under the natural condition

$$mes\{t : \omega(t) = 0\} = 0,$$

the fact that $\omega(t)\varphi_n(t) \in L_p$ for any n implies the completeness of $\{\omega(t)\varphi_n(t)\}$ in the space L_p. In addition, it is proved that this fact, whose analogues are well known for classical exponential and cosine systems, are, in general, not true for arbitrary complete or arbitrary complete orthonormal systems;

4. An example of a class of weight functions $\omega(t) \in L_p(0,\pi)$ is given for which the system $\{\omega(t)\cos nt\}_{n \in Z_+}$ is complete in $L_p(0,\pi)$, but neither this system, nor a system obtained by eliminating any finite number of its elements, is complete and minimal in $L_p(0,\pi)$;

5. Necessary condition for the basicity of Kostyuchenko type systems

$$\{\varphi^n(t)\sin nt\}_{n \in N},$$
$$\{\varphi^n(t)\cos nt\}_{n \in N \cup \{0\}}$$

in L_p $(1 \le p < +\infty)$ spaces is found; this condition implies, in particular, the necessary condition for the Kostyuchenko system

$$\{e^{i\alpha nt}\sin nt\}_{n \in \mathrm{N}}$$

to be a basis in L_p $(1 \le p < +\infty)$ space;

6. It is shown that a system of the form

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

is not a basis in the space $L_p[a,b]$, where $\varphi(t)$ is any measurable almost everywhere finite function on $[a,b]$;

7. It is shown that a system of the form

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

is not a basis in the space $C[a,b]$, where $\varphi(t)$ is any (real or complex-valued) continuous function on $[a,b]$;

8. It is shown that a system of the form

$$\{\varphi^n(t)\}_{n=0}^{\infty}$$

cannot be a pseudo-basis in $C[a,b]$, where $\varphi(t)$ is any (real or complex-valued) continuous function on $[a,b]$;

9. The frame properties of iterations

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

and

$$\{T_\varphi^n f\}_{n=-\infty}^{\infty}$$

of the multiplication operator

$$T_\varphi f(t) = \varphi(t) f(t), \quad f \in L_2(a,b)$$

in the space $L_2(a,b)$ are studied. It is shown that the orbit

$$\{T_\varphi^n f\}_{n=0}^{\infty}$$

of the multiplication operator T_φ cannot form a frame for the space $L_2(a,b)$ for any measurable generator $\varphi(t)$ and any $f \in L_2(a,b)$. All frames of the form $\{\varphi^n(t)\}_{n=-\infty}^{\infty}$ that are iterations of the multiplication operator T_φ are characterized. It is shown that this problem can be reduced to the following one: *Find (or describe a class) of all real functions* $\alpha(t)$, *for which*

$$\{e^{in\alpha(t)}\}_{n=-\infty}^{+\infty}$$

is a frame in $L_2(a,b)$. A partial answer to this problem is given.

10. Basicity properties of sequences of elements in a Hilbert space for which the angles between any two distinct elements are equal to the same nonzero number, and the basicity properties of sequences $\{x_n\}_{n=1}^{\infty}$ that have a bounded subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that

$$\left|\left(x_{n_k}, x_{n_m}\right)\right| = a > 0$$

for all sufficiently large $k, m \in N, k \neq m$ are studied;

11. A brief, simpler, and elementary proof of a characterization of compact operators in terms of orthonormal sequences is given; the proof of this fact presented in the dissertation shows that this fact is also true for operators acting from a Hilbert space into some Banach (which may not be Hilbert) space.

The main results of the dissertation are published in the following papers:

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