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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

İNVESTİGATİON SOLVABİLİTY PROBLEMS OF ELLIPTIC TYPE OPERATOR-DIFFERENTIAL EQUATIONS

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. It is known that the study of various problems for operator-differential equations is one of the most effective methods for the investigation of mixed problems for partial differential equations, infinite systems of ordinary differential equations and other problems. It should be noted that starting from the works of E.Hille, K.Iosida, T.Kato, S.Agmon, L.Nirinberg and Z.I.Khalilov, many mathematicians have studied the solution of the Cauchy problem for linear operator-differential equations with constant coefficients. Later, studies emerged on the possibility of solving boundary value problems for such equations. Most of these results were reflected in the books of E. Hille and R. Phillips, S.G.Krain, J.-L.Lions and E.Majenes, A.A.Dezin, V.I.Gorbachuk and M.L.Gorbachuk, S.Y.Yagubov and others. It should be noted that the theory of solving Cauchy and boundary value problems for the first and second order binomial operator-differential equations has been studied in detail by both foreign and Azerbaijani mathematicians. Later, many articles were published on high-order equations. Among these works include the articles of S.Agmon and L.Nirinberg, M.G.Gasimov, A.G.Kostyuchenko and A.A.Shkalikov, M.L.Gorbachuk, A.A.Dezin, S.Y.Yagubov, M.Bayramoglu, S.S.Mirzoyev, Q.V.Radziyevski, A.A.Shkalikov, A.B.Aliyev, V.V.Vlasov, H.D.Orucov, H.İ.Aslanov and A.R.Aliev. It should be noted that the main operator coefficient involved in the equations, or the main part of the equations studied in these articles, is the self-adjoint operator acting in a Hilbert space. Many practical problems require that the main operator coefficient should be from a large class when modeled by operator-differential equations. When modeling other practical problems with boundary value problems of operator-differential equations, there is a need for the participation of an abstract operator in boundary conditions. In other practical problems, equations have discontinuous coefficients. In this regard, both in advance and, in particular, recently, researches has been conducted in all three areas and relevant articles have been published. Among them the works L. de Simon and J. Torelli, A.A.Dezin, A.A.Shkalikov, V.V.Kornienko, S.Y.Yagubov and B.A.Aliyev, S.S.Mirzoyev, A.R.Aliev, A.M.Ahmadov and Y.Y.Mustafayeva, B.A.Aliyev and Y.Yakubov and their students can be noted. The factor that stimulated the research in most of these works was the proposed methodology in the articles of the academician M.G.Gasimov. In the future, this methodology will be developed in S.S.Mirzoev's articles and continued by A.R.Aliev's articles devoted to operator-differential equations with discontinuous coefficients.

It should be noted that, from the theoretical and practical point of view, the solution of abstract operator-differential equations (ie when all three factors are present at the same time) involving abstract operators in boundary conditions and having a normal operator in the main part of the equation is a great scientific interest. The presented dissertation work is mainly devoted to the problems of correct and unique solvability on the semiaxis for such elliptic-type operator-differential equations in Hilbert spaces. As mentioned above, here we use the methods developed in the works of S.S.Mirzoev and A.R.Aliev in the study of issues.

Object and subject of research. The object and subject of the dissertation research are the first and second boundary value problems on the semiaxis of the second-order elliptic operator-differential equations with discontinuous coefficients and a normal operator in the main part of the equation by the presence of abstract operators in boundary conditions.

Goal and tasks of the research. The main purpose and task of the dissertation find the conditions for solving the first and second boundary value problems of one class of second-order elliptic operator-differential equations with discontinuous coefficients in the presence of abstract operators in boundary conditions, to evaluate the norms of intermediate derivative operators in certain Sobolev-type vector functions spaces and consist of determining the relationship with the solution conditions.

Research methods. The methods of functional analysis, especially the analytical theory of operators with and without self-adjoining, the theory of linear operators in the Hilbert space, the theory of subgroups of operators, and the theory of generalized functions, were used in the dissertation work. In addition, the methods of the theory of differential equations in abstract spaces are also applied in the work.

The basic aspects to be defended. The following main statements are defended:

1. To find the conditions for solving the first boundary value problem on the semiaxis, one class of second-order elliptic operator-differential equations with discontinuous coefficient in the case of the bounded operator is presented in the boundary condition.

2. To find the conditions for solving the second boundary value problem on the semiaxis, one class of second-order elliptic operator-differential equations with discontinuous coefficient in the case of an unbounded operator is presented in the boundary condition.

3. To evaluate the norms of intermediate derivative operators in the spaces of certain Sobolev type vector functions, whose norms are written in operator-differential expression, and to determine their relation to the solvability conditions of the studied boundary value problems.

4. Investigate the properties of solutions of homogeneous second-order elliptic operator-differential equations with discontinuous coefficients on the semiaxis.

Scientific novelty of the research. In the dissertation work the following scientific innovations were obtained:

1. Correct and unique solvability conditions are found for the first boundary value problem on the semiaxis, one class of second-order elliptic operator-differential equations with discontinuous coefficient in the case of the bounded operator presented in the boundary condition.

2. Correct and unique solvability conditions are found for the second boundary value problem on the semiaxis, one class of second-order elliptic operator-differential equations with discontinuous coefficient in the case of an unbounded operator presented in the boundary condition.

3. The norms of intermediate derivative operators are evaluated in the spaces of certain Sobolev type vector functions, whose norms are written with operator-differential expression.

4. The relationship of the conditions of correct and unique solvability of boundary problems investigated with the evaluation of the norms of intermediate derivative operators has been determined.

5. The theorems have been proved on the correct and unique solvability on the semiaxis, one class of second-order homogeneous operator-differential equations with discontinuous coefficients within different non-homogeneous boundary conditions with operator coefficients.

6. The property of internal compactness of the space of regular solutions of second-order homogeneous elliptic operator-differential equations with discontinuous coefficient on the semiaxis is studied.

Theoretical and practical significance of the research. The work is mainly theoretical in nature. However, the results obtained in the dissertation can be used in a number of problems considered in the non-homogeneous environment of mathematical physics and in mechanics, for example, in the theory of elasticity for multilayer bodies.

Approbation and application. The results of the dissertation work were reported at the seminar of the department of "Mathematical Analysis" of Baku State University (head: Prof.S.S.Mirzoev), at the seminar of the department of "Theory of Functions and Functional Analysis" (head: Prof.A.M.Ahmadov), at the seminar of the department of "Applied Mathematics" (head: prof. H.D.Orujov), at the seminar of the department "Differential and integral equations" (head: prof. N.Sh.Isgandarov), at the seminar of the department of "Functional analysis" of the Institute of Mathematics and Mechanics of ANAS (head: prof.H.I.Aslanov), at the seminars of the "General and Applied Mathematics" department of the Azerbaijan State Oil and Industry University (head: prof.A.R.Aliyev). In addition, the results obtained in the dissertation were reported at the following scientific conferences: International Scientific Conference "Non-Newtonian systems in the oil and gas industry" dedicated to the 85th anniversary of academician A.X.Mirzajanzade at the Institute of Mathematics and Mechanics of ANAS (Baku, November 21-22, 2013) , at the International Scientific Conference "Spectral Theory of Differential Operators" dedicated to the 75th anniversary of academician M.G.Gasimov at the Institute of Mathematics and Mechanics of ANAS (Baku, December 8-10, 2014), at the Republican scientific conference "Actual problems of mathematics and mechanics" dedicated to the 92nd anniversary of national leader of the Azerbaijani people Heydar Aliyev at the Faculty of Mechanics and Mathematics of Baku State University (Baku, May 20-21, 2015), at a scientific International conference "Theoretical and applied problems of mathematics" dedicated to the 55th anniversary of Sumgayit State University (Sumgayit, May 25-26, 2017), at the International Scientific Conference "Operators, Functions and Systems in Mathematical Physics" dedicated to the 70th anniversary of prof.H.A.Isaxanli (Baku, May 21-24, 2018), at the IX International Youth Scientific and practical Conference at the Sterlitamak branch of Bashkir State University (Sterlitamak, October 30 - November 1, 2019).

Personal contribution of the author. All results and suggestions obtained belong to the author.

Author's publications. According to the research, 5 articles (1 is included in the SCIE list of Web of Science Core Collection, 1 is included in SCOPUS, 2 are included in Zentralblatt MATH databases), 1 conference material and 4 theses (in total 10 works) were published in the publishing houses recommended by the HCC under the President of the Republic of Azerbaijan. The list of works is given at the end of the abstract.

Name of the organization where the dissertation work is carried out. The work was performed at the deparment of "Theory of functions and functional analysis" of Baku State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately) General volume of the dissertation work consists of - 167292 sings (title page - 354 signs, table of contents - 2264 signs, introduction - 41212 signs, the first chapter - 50000 signs, the second chapter -

36000 signs, the third chapter - 36000 signs, conclusion - 1462). Then list of references 73 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, three chapters and a list of references.

In the introduction of the dissertation, the relevance of the topic is substantiated and the degree of its development is indicated, the goals and objectives of the research are stated, scientific novelty is given, theoretical and practical significance is noted, as well as information about the approbation of the work is given.

The first chapter is devoted to the study of the solution of the first boundary value problem on the semiaxis of second-order elliptic operator-differential equations with discontinuous coefficient. There is a normal operator with certain properties in the main part of the studied operator-differential equations and a bounded operator in the boundary condition. The theorems on the correct and unique solvability of these equations in a certain subspace of the second-order Sobolev space are proved.

Let H be a separable Hilbert space, A is a normal operator with an inverse operator in H.

Obviously, $D(A) = D(A^*)$, $A^*A = AA^*$ and the operator A is representable in the form $A = UC = CU$, here U is a unitary operator, C is a self-adjoint positive definite operator in H, so that $D(C) = D(A)$, and

$$\|Ax\| = \|A^*x\| = \|Cx\|, x \in D(A),$$
$$\|A^\gamma x\| = \|C^\gamma x\|, x \in D(A^\gamma) = D(C^\gamma), \gamma \geq 0.$$

It is known that, the domain of the operator C^γ $(\gamma \geq 0)$ will be H Hilbert space with scalar product:

$$H_\gamma = D(C^\gamma), (x,y)_\gamma = (C^\gamma x, C^\gamma y), x, y \in H_\gamma .$$

If $\gamma = 0$ then we assume that , $H_0 = H$ and

$$(x,y)_0 = (x,y),\ x,y \in H.$$

Let us denote by $L(X,Y)$ a space of linear bounded operators acting from the space X to space Y.

Let, $-\infty \le a < b \le \infty$. Denote by $L_2((a,b);H)$ set of all vector functions with the values in H, strongly measured and having finite integral:

$$\int_a^b \|f(t)\|_H^2 dt < \infty.$$

Note that $L_2((a,b);H)$ be a Hilbert space with the scalar product and norm as follows:

$$(f,g)_{L_2((a,b);H)} = \int_a^b \big(f(t),g(t)\big)dt,$$

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f(t)\|_H^2 dt\right)^{1/2}.$$

Consider $n \ge 1$, $u(t) \in L_2((a,b);H_n)$ vector functions and $u^{(n)}(t) \in L_2((a,b);H)$ are their generalized derivatives. Then the following linear set can be defined:

$$W_2^n\big((a,b);H\big) =$$
$$= \left\{u(t)\colon C^n u(t) \in L_2\big((a,b);H\big), u^{(n)}(t) \in L_2((a,b);H)\right\}.$$

Let us determine the scalar product in the $W_2^n\big((a,b);H\big)$ linear set as follows:

$$(u,\vartheta)_{W_2^n((a,b);H)} = (u^{(n)},\vartheta^{(n)})_{L_2((a,b);H)} + (C^n u, C^n \vartheta)_{L_2((a,b);H)},$$

here $u,\vartheta \in W_2^n\big((a,b);H\big)$. We obtain $W_2^n\big((a,b);H\big)$ a complete Hilbert space. It is clear that

$$\|u\|_{W_2^n((a,b);H)} = \left(\left\|u^{(n)}\right\|^2_{L_2((a,b);H)} + \|C^n u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

This norm can also be written as follows

$$\|u\|_{W_2^n((a,b);H)} = \left(\left\|u^{(n)}\right\|^2_{L_2((a,b);H)} + \|A^n u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

First, the definition of studied the regular solution of the second-order elliptic operator-differential equation with discontinuous coefficient on positive semiaxis $R_+ = [0;+\infty)$ and the definition of studied the regular solution of the first boundary value problem for this equation are given.

Consider in separable Hilbert space H the boundary value problem

$$-\frac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_1\frac{du(t)}{dt} + +A_2u(t) = f(t),\ t \in R_+ \quad (1)$$

$$u(0) = Tu'(0), \quad (2)$$

here $f(t), u(t)$ vector functions with the values H defined in R_+, the coefficients of the boundary condition of the equation satisfy the following conditions:

1) A is a normal operator with completely inverse A^{-1} and its spectrum is contained in the angular sector ;

$$S_\varepsilon = \left\{\lambda: |arg\lambda| \le \varepsilon,\ \ 0 \le \varepsilon < \frac{\pi}{2}\right\}$$

2) $\rho(t) = \begin{cases} \alpha^2, & t \in (0,1), \\ \beta^2, & t \in (1, +\infty), \end{cases}$ $\alpha, \beta > 0$ (assume for certainty that $\alpha \le \beta$);

3) $T \in L\left(H_{1/2}\ ,\ H_{3/2}\right)$, that is, the operator T is a linear continuous operator acting from space $H_{1/2}$ to space $H_{3/2}$;

4) A_1 and A_2 are linear operators such that, $B_1 = A_1A^{-1}, B_2 = A_2A^{-2}$ are bounded operators in H.

Definition 1. If $f(t) \in L_2(R_+; H)$ there exists a vector-function $u(t) \in W_2^2(R_+; H)$ satisfying equation (1) almost everywhere in R_+, then we say that it is a regular solution of equation (1).

Definiton 2. If for any $f(t) \in L_2(R_+; H)$ there exists a regular solution satisfying equation (1) and boundary condition (2) in the sense of convergence $\lim\limits_{t \to +0} \|u(t) - Tu'(t)\|_{3/2} = 0$ and it holds the estimation $\|u\|_{W_2^2(R_+;H)} \le const\|f\|_{L_2(R_+;H)}$, then boundary value problem (1), (2) is called regularly solvable.

Let's look at a subspace in $W_2^2(R_+; H)$ space:

$$W_{2,T}^2(R_+; H) = \{u: u \in W_2^2(R_+; H), u(0) = Tu'(0)\}.$$

Let us write the boundary value problem (1), (2) as follows

$$P_Tu = P_{0,T}u + P_{1,T}u = f\ , \quad (3)$$

here $f \in L_2(R_+;H)$, $u \in W_{2,T}^2(R_+;H)$,

$$P_{0,T}u = -\frac{d^2u}{dt^2} + \rho(t)A^2u,\ \ u \in W_{2,T}^2(R_+;H),$$

and

$$P_{1,T}u = A_1\frac{du}{dt} + A_2u,\quad u \in W_{2,T}^2(R_+;H).$$

The following statements are valid.

Lemma 1. *Let conditions 1)-3) be fulfilled. Then* $P_{0,T}$ *bounded operator from the space* $W_{2,T}^2(R_+;H)$ *to the space* $L_2(R_+;H)$.

Theorem 1. *Let conditions 1)-3) be fulfilled, and the operator*

$$Q_{\alpha,\beta} = E - \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A} + \alpha AT(E + \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A})$$

be inversible in $H_{1/2}$. *Then the operator* $P_{0,T}$ *realizes isomorphism from the space* $W_{2,T}^2(R_+;H)$ *onto the space* $L_2(R_+;H)$.

This theorem implies

Corollary 1. *The norms* $\|u\|_{W_{2,T}^2(R_+;H)}$ *and* $\|P_{0,T}u\|_{L_2(R_+;H)}$ *are equivalent in the space* $W_{2,T}^2(R_+;H)$.

Corollary 2. *The norms*

$$N_1(T) = \sup_{0\neq u\in W_{2,T}^2(R_+;H)} \|Au'\|_{L_2(R_+;H)}\|P_{0,T}u\|_{L_2(R_+;H)}^{-1}$$

and

$$N_2(T) = \sup_{0\neq u\in W_{2,T}^2(R_+;H)} \|A^2u\|_{L_2(R_+;H)}\|P_{0,T}u\|_{L_2(R_+;H)}^{-1}$$

are finite.

This raises the issue of estimating the norms of intermediate derivative operators.

Theorem 2. *Let conditions* 1)-3) *be fulfilled and* $ReUAT \geq 0$ *and* $ReCT \geq 0$ *in* $H_{1/2}$. *Then for any* $u(t) \in W_{2,T}^2(R_+;H)$ *the following inequalities hold:*

$$\|Au'\|_{L_2(R_+;H)} \leq c_1(\varepsilon)\|P_{0,T}u\|_{L_2(R_+;H)},$$

$$\|A^2u\|_{L_2(R_+;H)} \leq c_2(\varepsilon)\|P_{0,T}u\|_{L_2(R_+;H)},$$

here

$$c_1(\varepsilon) = \frac{1}{2\alpha}\frac{1}{\cos\varepsilon}\ ,\quad 0 \le \varepsilon < \frac{\pi}{2},$$
$$c_2(\varepsilon) = \begin{cases} \dfrac{1}{\alpha^2}\ ,\ \ 0 \le \varepsilon \le \dfrac{\pi}{4}\ , \\ \dfrac{1}{\sqrt{2}\alpha^2}\dfrac{1}{\cos\varepsilon}\ ,\quad \dfrac{\pi}{4} \le \varepsilon < \dfrac{\pi}{2}\ . \end{cases}$$

Two important remarks can be made about this theorem.

Remark 1. *It is easy to show that T can be chosen in such a way that both dissipativity conditions in theorem 2 are fulfilled.*

Remark 2. *Both of dissipativity conditions of the operators CT and UAT are independent.*

Finally, we learn the solution of equation (3).

The following statement is true.

Lemma 2. *Let conditions* 1)-4) *be fulfilled. Then $P_{0,T} + P_{1,T}$ is bounded operator from the space $W_{2,T}^2(R_+;H)$ to the space $L_2(R_+;H)$.*

Now, using the above results, we can state the main theorem of the first chapter.

Theorem 3. *Let conditions* 1)-4) *be fulfilled, the operator $Q_{\alpha,\beta}$ be inversible in $H_{1/2}$, $ReUAT \ge 0$* in *$H_{1/2}$, and $ReCT \ge 0$. In addition, the following condition is satisfied for operators B_1* and *B_2:*
$$c_1(\varepsilon)\|B_1\| + c_2(\varepsilon)\|B_2\| < 1.$$
Here $c_1(\varepsilon), c_2(\varepsilon)$ are defined in *teorem* 2. *Then boundary value problem* (1), (2) *is regularly solvable.*

Note the following two conclusions drawn from this theorem.

Corollary 3. *Let A be a self-adjoint positive operator, $T = 0$ and conditions* 2), 4) *be fulfilled. Then subject to the inequality*
$$2^{-1}\alpha^{-1}\|B_1\| + \alpha^{-2}\|B_2\| < 1$$
boundary value problem (1), (2) *is regularly solvable.*

Corollary 4. *Let conditions* 1), 3), 4) *be fulfilled, and $\rho(t) \equiv 1$, $t \in R_+$, $ReUAT \ge 0$ and $ReCT \ge 0$* in *$H_{1/2}$ and it hold the inequality:*
$$\tilde{c}_1(\varepsilon)\|B_1\| + \tilde{c}_2(\varepsilon)\|B_2\| < 1,$$
here

$$\tilde{c}_1(\varepsilon) = \frac{1}{2\cos\varepsilon}, \qquad 0 \le \varepsilon < \frac{\pi}{2},$$
$$\tilde{c}_2(\varepsilon) = \begin{cases} 1, & 0 \le \varepsilon \le \frac{\pi}{4}, \\ \frac{1}{\sqrt{2}\cos\varepsilon}, & \frac{\pi}{4} \le \varepsilon < \frac{\pi}{2} \end{cases}.$$

Then boundary value problem (1), (2) *is regularly solvable.*

Note that for $\varepsilon = 0$, $\rho(t) = 1, T = 0$ boundary value problem (1.1), (1.2) was first studied in M.G.Gasymov's papers[12]. The results obtained in these papers were generalized for some $T \in L\big(H_{1/2}\,,\ H_{3/2}\big)$ in the paper of M.G.Gasymov and S.S.Mirzoev[3]. For $\varepsilon = 0, T = 0$ and $\rho(t) \neq 1$ satisfying condition 2), problem (1), (2) was studied in the paper of S.S.Mirzoev and A.R.Aliev[4]. The analog of this case for higher order equations was investigated in the papers of A.R.Aliev[5], S.S.Mirzoev and A.R.Aliev[6]. In the sequel, for $\varepsilon = 0$, $\rho(t) = 1, T = 0$ boundary value problem (1), (2) was considered in

[1] Gasymov, M.G. The multiple completeness of part of the eigen- and associated vectors of polynomial operator bundles // The multiple completeness of part of the eigen- and associated vectors of polynomial operator bundles, – 1971. v. 6, no. 2–3, – pp. 131–147 (in Russian).

[2] Gasymov, M.G. On the theory of polynomial operator pencils // Soviet Mathematics Doklady, – 1972. vol. 12, – p. 1143–1147.

[3] Gasymov M.G., Mirzoev, S.S. Solvability of boundary value problems for second-order operator-differential equations of elliptic type // Differential Equations, – 1992. vol. 28, no. 4, – p. 528–536.

[4] Mirzoev, S. S., Aliev, A. R. On a boundary value problem for second-order operator-differential equations with a discontinuous coefficient // Proc. Inst. Math. Mech. Acad. Sci. Azerb., – 1997. vol. 6(14), – pp. 117–121 (in Russian).

[5] Aliev, A.R. Boundary-value problems for a class of operator differential equations of high order with variable coefficients // Mathematical Notes, – 2003. vol. 74, no. 5-6, – p. 761–771.

[6] Aliev, A.R., Mirzoev, S.S. On boundary value problem solvability theory for a class of high-order operator-differential equations // Functional Analysis and Its Applications, – 2010. vol. 44, no. 3, – p. 209–211.

the paper of S.S.Mirzoev, A.R.Aliev and L.A.Rustamova[7].

Note that in the present paper $\varepsilon \neq 0, T \neq 0$ and $\rho(t) \neq 1$ and this problem was not studied even in the case $\rho(t) = 1$. Therefore, the obtained results are new in the case $\rho(t) = 1$ as well.

The second chapter is devoted to the study of the solution of the second boundary value problem of the elliptic operator-differential equations with discontinuous coefficient on the semiaxis. Here, too, the operator involved in the main part of the operator-differential equations is a normal operator with certain properties, and there is an unbounded operator in the boundary condition. The main results are the derivation of theorems on the correct and unique solvability of these equations in a certain subspace of the second order Sobolev space.

In this chapter operator-differential equation in the form (1) is studied with the following boundary condition:

$$u'(0) = Ku(0), \tag{4}$$

Here, the same $f(t), u(t) \in L_2(R_+; H)$ vector functions with the values H defined in R_+, the coefficients of the boundary condition of the equation satisfy the following conditions:

1*) A is a normal operator with completely inverse A^{-1} and its spectrum is contained in the angular sector ;

$$S_\varepsilon = \left\{\lambda: |arg\lambda| \leq \varepsilon,\ \ 0 \leq \varepsilon < \frac{\pi}{2}\right\}$$

2*) $\rho(t) = \begin{cases} \alpha^2, & t \in (0,1), \\ \beta^2, & t \in (1, +\infty), \end{cases}$ $\alpha, \beta > 0$ (assume for certainty that $\alpha \leq \beta$);

3*) $K \in L\left(H_{1/2}\ ,\ H_{3/2}\right)$, that is, the operator K is a linear continuous operator acting from space $H_{3/2}$ to space $H_{1/2}$;

4*) A_1 and A_2 are linear operators such that, $B_1 = A_1 A^{-1}, B_2 = A_2 A^{-2}$ are bounded operators in H.

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[7] Mirzoev, S.S., Aliev, A.R., Rustamova, L.A. On the boundary value problem with the operator in boundary conditions for the operator-differential equation of second order with discontinuous coefficients // Journal of Mathematical Physics, Analysis Geometry, – 2013. vol. 9, no. 2, – p. 207–226.

Definiton 3. If for any $f(t) \in L_2(R_+; H)$ there exists a regular solution satisfying equation (1) and boundary condition (2) in the sense of convergence $\lim\limits_{t \to +0} \|u'(t) - Ku(t)\|_{1/2} = 0$ and it holds the estimation $\|u\|_{W_2^2(R_+;H)} \le const\|f\|_{L_2(R_+;H)}$, then boundary value problem (1), (2) is called regularly solvable.

First, lets denote with $\mathrm{P}_{0,K}$ operator from the space $\mathrm{W}_{2,\mathrm{K}}^2(\mathrm{R}_+; \mathrm{H})$ to the space $L_2(R_+; H)$:

$$P_{0,K}u(t) = -\frac{d^2u}{dt^2} + \rho(t)A^2u(t), \quad u(t) \in W_{2,K}^2(R_+; H).$$

Note that here $W_{2,K}^2(R_+; H)$ is the subspace of the space $W_2^2(R_+; H)$:

$$W_{2,K}^2(R_+; H) = \{u: u \in W_2^2(R_+; H), u'(0) = Ku(0)\}.$$

The following statements are true.

Lemma 3. *Let conditions 1*)-3*) be fulfilled. Then* $P_{0,K}$ *bounded operator from the space* $W_{2,K}^2(R_+; H)$ *to the space* $L_2(R_+; H)$.

Theorem 4. *Let conditions 1*)-3*) be fulfilled, and the operator*

$$R_{\alpha,\beta} = E + \frac{\beta - \alpha}{\beta + \alpha}e^{-2\alpha A} + \frac{1}{\alpha}A^{-1}K(E - \frac{\beta - \alpha}{\beta + \alpha}e^{-2\alpha A})$$

be inversible in $H_{3/2}$. *Then the operator* $P_{0,K}$ *realizes isomorphism from the space* $W_{2,K}^2(R_+; H)$ *onto the space* $L_2(R_+; H)$.

This theorem implies

From theorem 4 also yields corresponding results such as corollary 1 and corollary 2 in the first chapter. For this reason, the second chapter also raises the issue of estimating the following quantities:

$$N_1(K) = \sup_{0 \ne u \in W_{2,K}^2(R_+;H)} \|Au'\|_{L_2(R_+;H)} \left\|P_{0,K}u\right\|_{L_2(R_+;H)}^{-1},$$

$$N_2(K) = \sup_{0 \ne u \in W_{2,K}^2(R_+;H)} \|A^2u\|_{L_2(R_+;H)} \left\|P_{0,K}u\right\|_{L_2(R_+;H)}^{-1}.$$

The following theorem is valid.

Theorem 5. *Let conditions 1*)-3*) be fulfilled, the operator $R_{\alpha,\beta}$ be inversible in $H_{3/2}$ and $ReU^{-1}A^{-1}K \geq 0$, $ReC^{-1}K \geq 0$* in $H_{3/2}$. Then the following inequalities are valid:

$$N_1(K) \leq \frac{1}{2\alpha\cos\varepsilon}\ , \qquad 0 \leq \varepsilon < \frac{\pi}{2},$$

$$N_2(K) \leq \frac{1}{\alpha^2}\begin{cases} 1, & 0 \leq \varepsilon \leq \dfrac{\pi}{4}\ , \\ \dfrac{1}{\sqrt{2}\cos\varepsilon}, & \dfrac{\pi}{4}\ \leq \varepsilon < \dfrac{\pi}{2}\ . \end{cases}$$

Finally, let's move on to the theorem on the regular solution of the boundary value problem (1), (4) when $A_j \neq 0, j = 1,2$.

Lemma 4. *Let conditions* 1*)-4*) *be fulfilled. Then $P_{0,K} + P_{1,K}$ is bounded operator from the space $W^2_{2,K}(R_+;H)$ to the space $L_2(R_+;H)$.*

Theorem 6. *Let conditions* 1*)-4*) *be fulfilled, the operator $R_{\alpha,\beta}$ be inversible in $H_{3/2}$, $Re\ U^{-1}A^{-1}K \geq 0$* in $H_{3/2}$, $ReC^{-1}K \geq 0$ and *the following condition is satisfied for operators* B_1 and B_2:

$$N_1(K)\|B_1\| + N_2(K)\|B_2\| < 1.$$

Here $N_1(K)$, $N_2(K)$ are defined in *teorem* 5. *Then boundary value problem* (1), (2) *is regularly solvable.*

In practical problems, it is difficult to verify both conditions of the operators $Q_{\alpha,\beta}$ and $R_{\alpha,\beta}$ are participating in the statements of the first chapter and second chapter have an unbounded inverse operator in $H_{1/2}$ and $H_{3/2}$ respectively. Note that, for example, satisfying the condition $ReA^{-1}K \geq 0$ in $H_{3/2}$ provides that the operator $R_{\alpha,\beta}$ has an inverse in $H_{3/2}$. Therefore, in the statements of the second chapter, instead of having the inverse of the operator $R_{\alpha,\beta}$, it is possible to demand that the condition $ReA^{-1}K \geq 0$ be satisfied in $H_{3/2}$. This condition, in turn, is checked more easily than the condition that the operator $R_{\alpha,\beta}$ has the inverse in applied problems.

Let us note the following two results from the theorem 6 at the end of the second chapter

Corollary 5. *Let A be a self-adjoint positive operator, $K \in L(H_{3/2}, H_{1/2})$ and $ReA^{-1}K \geq 0$* in *$H_{3/2}$. Then subject to the inequality*

$$2^{-1}\alpha^{-1}\|B_1\| + \alpha^{-2}\|B_2\| < 1$$

boundary value problem (1), (4) *is regularly solvable.*

Corollary 6. *Let A be a normal operator with completely continuous inverse A^{-1} and spectrum contained in an angular sector S_ε,* suppose that *$ReA^{-1}K \geq 0$, $ReC^{-1}K \geq 0$ and $ReU^{-1}A^{-1}K \geq 0$* in *$H_{3/2}$, and the following inequality is valid:*

$$c_1(\varepsilon)\|B_1\| + c_2(\varepsilon)\|B_2\| < 1,$$

here

$$c_1(\varepsilon) = \frac{1}{2\cos\varepsilon}, \quad 0 \leq \varepsilon < \frac{\pi}{2},$$
$$c_2(\varepsilon) = \begin{cases} 1, & 0 \leq \varepsilon \leq \frac{\pi}{4}, \\ \frac{1}{\sqrt{2}\cos\varepsilon}, & \frac{\pi}{4} \leq \varepsilon < \frac{\pi}{2}. \end{cases}$$

Then the boundary value problem (1), (4) *is regularly solvable for $\rho(t) = 1, t \in R_+$.*

In our opinion, as in the first chapter, the last result in this chapter is new for $\rho(t) = 1, t \in R_+$.

Note that (1), (4) the boundary value problem was studied in the papers of M.G.Gasymov and S.S.Mirzoev[3], S.S.Mirzoev and Kh.V.Yagubova[8] when operator A is the self-adjoint, positive-definite operator and $\alpha = \beta = 1$. In addition, when $K = 0$ and A is a normal operator then the regular solution of the boundary value problem (1), (4) was considered in the work S.S.Mirzoev and L.A.Rustamova[9]. In the work S.S.Mirzoev, A.R.Aliev and

[8] Mirzoev, S. S., Yagubova, Kh. V. On the solvability of boundary value problems with operators in the boundary conditions for a class of second-order operator-differential equations // Dokl. Nat. Akad. Nauk Azerb., – 2001. vol. 57, no. 1–3, – pp. 12–17 (in Russian).

[9] Mirzoev, S.S., Aliev, A.R., Rustamova, L.A. Solvability conditions for boundary-value problems for elliptic operator-differential equations with discontinuous coefficient // Mathematical Notes, – 2012. vol. 92, no. 5-6, – p. 722–726.

L.A.Rustamova[10] the cases A is a self-adjoint, positive-definite operator and $A_2 = 0$ were investigated.

The third chapter examines the regular solution of the second-order homogeneous operator-differential equation with discontinuous coefficients within various non-homogeneous boundary conditions with operator coefficients. In addition, the property of internal compactness of the space of regular solutions of the given equation is studied.

In this chapter considered the following boundary problem in the H separabel Hilbert space:

$$P\left(\frac{d}{dt}\right)u(t) \equiv -\frac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_1\frac{du(t)}{dt} + \\ +A_2u(t) = 0,\ t \in R_+ , \qquad (5)$$

$$u(0) - Tu'(0) = \varphi, \qquad (6)$$

here, $u(t)$ is the vector function with the values H defined in R_+, the coefficients of the boundary condition of the equation satisfy the following 1)-4) conditions in the first chapter.

Definition 4. If there exists a vector-function $u(t) \in W_2^2(R_+; H)$ satisfying equation (5) almost everywhere in R_+, then we say that it is a regular solution of equation (5).

Definiton 5. If for any $\varphi \in H_{3/2}$ there exists a regular solution satisfying equation (5) and boundary condition (6) in the sense of convergence $\lim\limits_{t\to+0}\|u(t) - Tu'(t) - \varphi\|_{3/2} = 0$ and it holds the estimation $\|u\|_{W_2^2(R_+;H)} \le const\|\varphi\|_{3/2}$, then boundary value problem (5), (6) is called regularly solvable.

The theorem on the regular solvability of the boundary value problem (5), (6) is proved by using the results obtained in the first chapter

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[10] Mirzoev, S.S., Aliev, A.R., Rustamova, L.A. Solvability conditions for boundary-value problems for elliptic operator-differential equations with discontinuous coefficient // Mathematical Notes, – 2012. vol. 92, no. 5-6, – p. 722–726.

Theorem 7. Suppose that all the conditions of theorem 3 are satisfied. Then boundary value problem (5), (6) is regularly solvable.

Analogy consider the coefficients of the homogeneous operator-differential equation (5) are satisfying 1), 2) and 4) conditions with

$$u'(0) - Ku(0) = \psi \tag{7}$$

non-homogeneous boundary condition in H separabel Hilbert space. Here $\psi \in H_{1/2}$, operator K is satisfying the condition 3*).

Definiton 6. If for any $\psi \in H_{3/2}$ there exists a regular solution satisfying equation (5) and boundary condition (7) in the sense of convergence $\lim_{t \to +0} \|u'(t) - Ku(t) - \psi\|_{1/2} = 0$ and it holds the estimation $\|u\|_{W_2^2(R_+;H)} \leq const\|\psi\|_{1/2}$, then boundary value problem (5), (7) is called regularly solvable.

We can state the theorem on the regular solvability of the boundary value problem (5), (7) by using the results obtained in the second chapter.

Theorem 8. Suppose that all the conditions of theorem 6 are satisfied. Then boundary value problem (5), (7) is regularly solvable.

Let denote with the $Ker(P\left(\frac{d}{dt}\right), R_+)$ set of solutions of equation (5) from the space $W_2^2(R_+; H)$:

$$Ker\left(P\left(\frac{d}{dt}\right), R_+\right) = \left\{u(t): P\left(\frac{d}{dt}\right)u(t) = 0, u(t) \in W_2^2(R_+; H)\right\}.$$

In other words, $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ is a space of regular solutions of equation (5).

It is clear that the set $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ is a closed subspace of the space $W_2^2(R_+; H)$.

Now let's denote with $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ closure of the space $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ respect to the norm of the space $W_2^1(R_+; H)$.

The internal compactness property of the $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ space is determined by the norm of $\|u\|_{W_2^1(R_+;H)}$ at the end of this chapter.

Definition 7. *Suppose that*

$$0 \leq a < a' < b' < b < +\infty$$

and $M > 0$ *is arbitrary number. If the set*

$$W_M = \left\{ u(t) : u(t) \in \mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right), \|u\|_{W_2^1((a,b);H)} \leq M \right\}$$

is compact set in the norm of space $W_2^1((a',b');H)$, *then we will say that* $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ *space is an internal compact space in the norm of space* $W_2^1(R_+;H)$.

The following theorem is valid.

Theorem 9. *Let conditions* 1), 2), 4) *be fulfilled and for a vector function* $u(t)$ *from space* $\dot{W}_2^2(R_+;H)$ *the following inequality hold:*

$$\left\| P\left(\frac{d}{dt}\right) u \right\|_{L_2(R_+;H)} \geq const \|u\|_{W_2^2(R_+;H)} .$$

Then the space $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ *is an internal compact space and there is such a number that* $\varkappa_0 > 0$, *for any* $u(t) \in \mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ *the following inequality hold:*

$$\int_0^{+\infty} e^{-2\varkappa_0 t} \left(\left\| \frac{du(t)}{dt} \right\|_H^2 + \|Cu(t)\|_H^2 \right) dt < +\infty.$$

Here $\dot{W}_2^2(R_+;H) \subset W_2^2(R_+;H)$ and define as follows:

$$\dot{W}_2^2(R_+;H) = \{u(t) : u(t) \in W_2^2(R_+;H), u(0) = u'(0) = 0\},$$

C is self-adjoint positive-definite operator in H.

It should be noted that for the first time P.D.Lax gave the definition of internal compactness for some solution spaces in infinite range and showed its close connection with the Phragmen-Lindelof principle for solutions of elliptic equations.

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CONCLUSİON

The dissertation is devoted to the problems on the semiaxis of correct and unique solvability of elliptic type operator-differential equations with discontinuous coefficients involving abstract operators in boundary conditions and having a normal operator in main part of equation.

The main results of the dissertation are as follows:

1. Correct and unique solvability conditions are found for the first boundary value problem on the semiaxis of one class of second-order elliptic operator-differential equations with discontinuous coefficient and a normal operator in the main part of the equation in the case of a bounded operator present in the boundary condition.

2. Correct and unique solvability conditions are found for the second boundary value problem on the semiaxis of one class of second-order elliptic operator-differential equations with discontinuous coefficient and a normal operator in the main part of the equation in the case of an unbounded operator in the boundary condition.

3. The norms of intermediate derivative operators are evaluated in the spaces of certain Sobolev type vector functions, whose norms are written with operator-differential expression.

4. The relationship of the conditions of correct and unique solvability of boundary problems investigated with the evaluation of the norms of intermediate derivative operators has been determined.

5. The theorems have been proved on the correct and unique solvability on the semiaxis of one class of second-order homogeneous operator-differential equations with discontinuous coefficients and a normal operator in the main part of the equation within different non-homogeneous boundary conditions with operator coefficients.

6. The property of internal compactness of the space of regular solutions of second-order homogeneous elliptic operator-differential equations with discontinuous coefficient and a normal operator in the main part of the equation on the semiaxis is studied.

The main results of the dissertation were published in the following works:

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2. Гасымова, Г.М. О краевой задаче для операторно-дифференциального уравнения второго порядка с оператором в краевом условии // Материалы Международной научной конференции "Неньютоновские системы в нефтегазовой отрасли", посвященной 85-летнему юбилею академика Азада Халил оглы Мирзаджанзаде, – Баку: – 21–22 ноября, – 2013, – с. 73–74.
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6. Mirzoev, S.S., Aliev, A.R., Gasimova, G.M. Solvability conditions of a boundary value problem with operator coefficients

and related estimates of the norms of intermediate derivative operators // Doklady Mathematics, – 2016. vol. 94, no. 2, – pp. 566–568. (**Web of Science Core Collection, SCIE**)

7. Gasimova, G.M. On solvability conditions of a boundary value problem with an operator in the boundary condition for a second order elliptic operator-differential equation // – Baku: Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, – 2014. vol. 40, Special Issue, – p. 172–177. (**Zentralblatt MATH**)

8. Gasimova, G.M. On well-defined solvability of a boundary value problem for an elliptic differential equation in Hilbert space // – Baku: Transactions of National Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences, Issue Mathematics, – 2015. vol. 35, no. 1, – p. 31–36. (**SCOPUS**)

9. Gasimova, G.M. On a boundary-value problem for an equation with operator coefficients in a Hilbert space // Proceedings of an International conference "Operators, Functions and Systems of Mathematical Physics" dedicated to 70-th anniversary of prof. Hamlet Isakhanli, Khazar University, – Baku: – 21–24 May, – 2018, – p. 87–88.

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