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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVESTIGATION OF GLOBAL BIFURCATION OF NONLINEAR STURMİAN SYSTEMS OF FOURTH ORDER

Specialty: 1201.01 – Analysis and functional analysis

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic and degree of processing. One of the basic sections of modern analysis is theory of bifurcation of non-linear eigenvalue problems with indefinite (sign-chaning) weight. Such problems for second-order differential equations arise when modeling dynamics of population living in a highly heterogeneous environment, as well as selection-migration in population genetics. Problems of this type for fourth order differential equations arise when describing traveling waves in a hanging bridge, statical flexion of an elastic plate in liquid, image processing, the processes of filtration of barotropic gas through a porous medium.

Oscillation properties of eigenfunctions corresponding to linear spectral problems play an important role when studying bifurcation of solutions of nonlinear eigenvalue problems. As is known, the oscillatory properties of the eigenfunctions of the Sturm-Liouville problem were studied in detail back in the 30s of the 19th century by Sturm[1]. These properties of eigenfunctions of linear eigenvalue problems for ordinary differential equations fourth order (more precisely, for completely regular fourth-order Sturm systems) in the presence of potential were studied in detail in the recent paper of Z.S. Aliyev[2]. The oscillation properties of the eigenfunctions of the Sturm-Liouville problem with an indefinite weight function were studied by E.L. Ince[3] using the Sturm comparison theorem and Picone formulas. Moreover, in the works of G.A. Afrouzi and K.J. Brown, W. Allegretto, K.J. Brown and S.S. Lin, J. Fleckinger and M.L. Lapidus, P. Hess and T. Kato was proved the existence of principal eigenvalues (i.e., eigenvalues that correspond to positive or

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[1]Sturm, C. Sur une classe d'equations *a* derivee partielle//Journal de Mathematiques Pures et Appliquees, – 1836. v.1, – p. 373–444

[2]Aliyev, Z.S. Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, Sb. Math., - 2016, v. 207, no. 12, - p. 1625–1649

[3]Ince, E.L. Ordinary differential equations / E.L. Ince. – New York: Dover, – 1926, – 558 p.

negative eigenfunctions) of spectral problems for elliptic partial differential equations of second and fourth order with an sign-chaning weight function. Note that oscillatory properties of eigenfunctions of linear spectral problems for fourth order ordinary differential equations have not yet been studied.

Local and global bifurcation of solutions of nonlinear eigenvalue problems for ordinary and partial differential equations of second and fourth orders with a constant sign weight function were studied in detail in the papers of P.H. Rabinowitz[4,5], J.F. Toland, K.A. Stuart, A. Berestycki[6], P. Chiappinelli, J. Przybycin, B.P. Rynne, K. Schmidt and H.L. Smith, J. Chu, D. O'Regan, R.Ma and B. Tompson, A.S. Lazer and P.J. McKenna and others. These questions for second order differential equations with an sign-chaining weight function were studied by P. Hess and T. Kato, K.J. Brown, S.S. Lin and A. Terticas, W. Allegretto and A. Mingarelli, K.J. Brown, B. Ko and C. Brown, R.S. Cantrell and C. Cosner, Z.S. Aliyev, and Sh.M. Hasanova, Z.S. Aliyev and L.V. Nasirova and others. They have obtained global results on bifurcation of solutions that play an important role in population genetics.

Bifurcation of solutions of nonlinear eigenvalue problems for differential equations of fourth order with an indefinite weight function was studied in the papers of M. Delgado and A. Suarez, M.F. Furtado and J.P.P. da Silva, R. Ma, C. Gao and X. Han, E.D. Silva, J.C. de Albuquerque and T.R. Cavalcante, J. Wang and R.Ma. In these papers, the existence of global continua of solutions contained in the classes of positive and negative functions was proved. Since the oscillation properties of eigenfunctions of linear spectral problems for ordinary differential equations of fourth order with an indefinite weight function have not yet been studied, for this

[4] Rabinowitz, P.H. Some global results for nonlinear eigenvalue problems // J. Function. Anal., – 1971. v.7, no.3, – p. 487–513.
[5] Rabinowitz, P.H. On bifurcation from infinity // J. Diff. Equat. – 1973. v.14, no. 3, – p. 462–475
[6] Berestycki, H. On some nonlinear Sturm-Liouville problems// J.Diff. Equat., – 1977. v. 26, no. 3, – p. 375–390.

reason the local and global bifurcation of solutions to nonlinear eigenvalue problems for ordinary differential equations of fourth order with an indefinite weight function have not been studied.

Thus, the study of oscillation properties of eigenfunctions of linear spectral problems and the study of local and global bifurcation of solutions from zero and from infinity of nonlinear eigenvalue problems for ordinary differential equations of fourth order (i.e., for the fourth order Sturmian system) with an indefinite weight function is relevant.

Object and subject of the study. The object of study is linear and nonlinear eigenvalue problems for fourth-order ordinary differential equations with an indefinite weight function, and the subject of research is the oscillatory properties of eigenfunctions of linear spectral problems and the global bifurcation from zero and from infinity of a set of nontrivial solutions to nonlinear eigenvalue problems.

Goal and tasks of the study. The main goal of the dissertation is to study the oscillatory properties of eigenfunctions of linear problems and their derivatives, and to study the bifurcation of nontrivial solutions from zero and from infinity of nonlinear eigenvalue problems for ordinary differential equations of fourth order with indefinite weight.

İnvestigation methods. The dissertation work mainly uses methods of ordinary differential equations, function theory and functional analysis, differential geometry and topology, spectral theories of differential operators, bifurcation theory, and nonlinear functional analysis.

Basic statements to be defended. The following basic statements are to be defended:

– to prove the existence of infinitely increasing and infinitely decreasing sequence of positive and negative simple eigenvalues;

– to study oscillation properties of eigenfunctions of the linear spectral problem for fourth-order ordinary differential equations with an indefinite weight function;

– to study local and global bifurcation of solutions from zero and infinity of linearizable eigenvalue problems for fourth-order ordinary differential equations with an sign-changing weight;

– to study local and global bifurcation of positive and negative solutions from zero and infinity of nonlinear nonlinearizable eigenvalue problems for fourth-order ordinary differential equations with a sign-changing weight function.

Scientific novelty of the study. The followings are the basic results of the dissertation work:

– the existence of two infinitely increasing and infinitely decreasing sequences of positive and negative simple eigenvalues of a linear problem for a fourth-order Sturmian system with an indefinite weight function, was proved;

– oscillation properties of eigenfunctions corresponding both to positive and also negative principal eigenvalues of fourth order Sturmian system with an indefinite weight function were studied;

– the existence of four unbounded continua of solutions bifurcating from the points of the line of trivial solutions and $R \times \{\infty\}$, and contained in the classes of functions possessing oscillation properties of principal eigenfunctions and their derivatives of the corresponding linear problems was shown;

– the existence of four unbounded continua of solutions emanating from intervals of the line of trivial solutions and $R \times \{\infty\}$, and contained in the classes of positive and negative functions is proved.

Theoretical and practical value of the study. The results obtained in the dissertation work are mainly of theoretical character. These results can be applied in modeling electrorheological fluids and other phenomena related to image processing and flow in a porous medium, static deformation of an elastic plate in a fluid.

Approbation and application. The results obtained in the dissertation work were reported in the departments of «Non-harmonic analysis» (head: corr-memb. of ANAS, prof. B.T. Bilalov), «Differential equations» (head: prof. A.B. Aliyev) and «Functional anlaysis» (head: prof. H.I. Aslanov) of Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of

Azerbaijan, in the International Conference MADEA-7 of «Mathematical Analysis, Differential Equations and their Applications»(Baku, 2015), in the International Workshop «Non-harmonic analysis and differential operators» (Baku, IMM 2016), in the VIII Annual International Conference of Georgian Mathematical Society (Batumi, Georgia, 2017), in the Republican Scientific Conference "Theory of functions, functional analysis and their applications", dedicated to the 110th anniversary of the birth of academician Ibrahim Ibrahimov (Baku, BSU, 2022).

Author's personal contribution is in formulation of the goal of the study, in addition, all obtained results belong to the author.

Author's publications. The main results of the dissertation work were published in 5 scientific articles (3 of them WOS, 1 SCOPUS) in journals recommended by the Higher Attestation Commission under the President of the Republic of Azerbaijan and 4 materials of International Conferences (one of them was held abroad).

The institution where the dissertation work was performed. The dissertation work was carried out at the department of "Non-harmonic analysis" of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Structure and volume of the dissertation work (in signs, indicating the volume of each structural unit separately).

Total volume of the work- 212398 signs (title page-373 signs, content 2530 signs, introduction-58268 signs, chapter I -48000 signs, chapter II-52000 signs, chapter III- 50000, conclusion -1227). The list of references consists of 95 titles.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, 3 chapters, conclusion and a list of references.

Chapter I consisting of five sections was devoted to the study of oscillation properties of a spectral problem for fourth order ordinary differential equations with an indefinite weight function.

Section 1.1 outlines the formulation of the problem.

Consider the following spectral problem

$$(\tau(x)y''(x))'' = \lambda r(x)y(x),\ x \in (0, 1), \tag{1}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{2}$$

where $\lambda \in C$ is a spectral parameter, the function $\tau(x)$ has an absolutely continuous derivative and is positive on $[0, 1]$, the weight function $r(x)$ is continuous and changes sign on the interval $[0, 1]$.

The eigenvalue problem (1), (2) in the case of $r(x) > 0,\ x \in [0, 1],$ was studied by S.N. Janczewsky[7]. He has proved that all eigenvalues of this problem are positive, simple and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^{\infty}$. Moreover, the eigenfunction $\vartheta_k(x)$, corresponding to the k th eigenvalue μ_k, has exactly $k-1$ simple zeros in the interval $(0, 1)$.

It is known that oscillation properties of eigenfunctions of the Sturm-Liouville problem with an indefinite weight has been studied in detail in the mentioned work of E.L.Ince[3]. These properties for eigenfunctions of spectral problems for ordinary differential operators of fourth order have not been studied.

The goal of this chapter is to study spectral properties of boundary value problems for fourth-order ordinary differential equations with completely regular boundary conditions and sign-changing weight function. We will prove that these spectral problems have two infinitely increasing and infinitely decreasing sequences of positive and negative eigenvalues, respectively. Moreover, the smallest positive and largest negative eigenvalues are simple, and the corresponding eigenfunctions do not have zeros in the interval $(0, 1)$.

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[7] Janczewsky, S.N. Oscillation theorems for the differential boundary value problems of the fourth order // Ann. Math., – 1928. v. 29, no. 2, – p. 521-542

In 1.2 we give the results on critical points of smooth functionals in C^1-manifolds.

1.3 was devoted to proving the existence of infinitely many positive and negative eigenvalues of problem (1), (2).

Let $I=(0,1)$ and $W^{k,p,\tau}(I)$ be a Sobolev weight space consisting of all measurable real functions u determined on I and for which

$$\|u\|_{k,p,\tau}=\left\{\sum_{m=0}^{k-1}\int_I|u^{(m)}(x)|^p\,dx+\int_I\tau(x)|u^{(k)}(x)|^p\,dx\right\}^{\frac{1}{p}}<+\infty.$$

Let $W_0^{1,p}(I)$ be a closure of $C_0^{\infty}(I)$ in $W^{1,p}(I)=W^{1,p,1}(I)$.

Denote $X=W_0^{1,2}(I)\bigcap W_0^{2,2,\tau}(I)$ with the norm

$$\|u\|_X=\left\{\int_I\tau(x)|u''(x)|^2\,dx\right\}^{1/2},$$

that due to the Friedrichs weight inequality is equivalent to the norm $\|u\|_{2,2,\tau}$ of space $W^{2,2,\tau}(I)$. Then, by X^* we denote a space dual to the space X.

We introduce linear operators $L,H:X\to X^*$ as follows:

$$\langle L(u),\vartheta\rangle=\int_I\tau u''\vartheta''dx,$$

$$\langle H(u),\vartheta\rangle=\int_I\tau u\vartheta\,dx,\ \ u,\vartheta\in X.$$

We define the following functionals on X:

$$F(u)=\frac{1}{2}\int_I\tau|u''|^2dx,$$

$$G(u)=\frac{1}{2}\int_I r|u|^2dx.$$

Let

$$M=\{u\in X:2G(u)=1\}.$$

Then, problem (1), (2) for $\lambda>0$ can be written in the following equivalent form

$$L(u) = \lambda H(u), u \in M. \quad (4)$$

Note that (λ, u) is the solution of problem (3) (or (4)) if and only if u is a critical point of the functional F on the set M.

Let E be a real Banach space, Σ be a totality of all symmetric subsets $E \setminus \{0\}$, closed in E (the set $Y \subset E$ is called symmetric if $Y = -Y$). Following M.A. Krasnosel'skii[8] the nonempty set $Y \subset \Sigma$ is said to be of genus $k, k \in \mathrm{N} \bigcup \{0\}$, (denoted by $\gamma(Y) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping from Y to $R^k \setminus \{0\}$. If there is no such a k, then $\gamma(Y) = +\infty$, and if $Y = \varnothing$, then $\gamma(Y) = 0$.

Denote:

$$\Gamma_n = \{ K \subset M : K \text{ is symmetric, compact and } \gamma(K) \geq n \}.$$

Lemma 1. *For any $k \in \mathrm{N}$ we have the relation:*

$$\Gamma_k \neq \varnothing.$$

One of the main results of this chapter is the following theorem.

Theorem 1. *For each $k \in \mathrm{N}$ the number*

$$\lambda_k^+ = \inf_{K \in \Gamma_k} \max_{u \in K} 2F(u) > 0$$

is a critical value of the functional F on M more exactly, there exists such $u_k^+ \in K_k \in \Gamma_k$, that

$$\lambda_k^+ = 2F(u_k^+) = \sup_{u \in K_k} 2F(u)$$

and u_k^+ is an eigenfunction of problem (1), (2), corresponding to the positive eigenvalue λ_k^+. Furthermore, $\lambda_k^+ \to +\infty$ as $k \to \infty$.

Corollary 1. *We have the relation:*

$$\lambda_1^+ \leq \lambda_2^+ \leq \ \ldots \ \leq \lambda_k^+ \mapsto +\infty.$$

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[8] Красносельский, М.А. Топологические методы в теории нелинейных интегральных уравнений/ М.А. Красносельский. – Москва; Ленинград: Гос. издательство техн.–теорет. лит., – 1956. – 392 с.

Corollary 2. *Problem (1), (2) has an infinitely decreasing sequence of negative eigenvalues* $\{\lambda_k^-\}_{k=1}^{\infty}$ *such that* $\lambda_k^- \to -\infty$ as $k \to \infty$.

In section 1.4 we study the properties of principal eigenvalues of problem (1), (2).

Theorem 2. *The eigenvalue* λ_1^+ (λ_1^-) *is simple and corresponding eigenfunction* $u_1^+(x)$ $(u_1^-(x))$ *does not vanish in the interval* I.

In 1.5 we consider a completely regular Sturm system with an indefinite weight function.

Let us consider the following eigenvalue problem for fourth order ordinary differential equations

$$\ell(u) \equiv (\tau(x)u'')'' - (q(x)u')' = \lambda r(x)u,\ 0 < x < 1, \tag{5}$$

$$u'(0)\cos\alpha - (\tau u'')(0)\sin\alpha = 0, \tag{6a}$$

$$u(0)\cos\beta + Tu(0)\sin\beta = 0, \tag{6b}$$

$$u'(1)\cos\gamma + (\tau u'')(1)\sin\gamma = 0, \tag{6c}$$

$$u(1)\cos\delta - Tu(1)\sin\delta = 0, \tag{6d}$$

where $q-$is a non-negative absolutely continuous function on the interval $[0, 1]$,, while $\alpha, \beta, \gamma, \delta$ are real constants such that $\alpha, \beta, \gamma, \delta \in [0, \pi/2]$, except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$ for $q \equiv 0$.

Theorem 3. *Spectral problem (5), (6) has two sequence of real eigenvalues*

$$0 < \lambda_1^+ \le \lambda_2^+ \le \ \ldots \ \le \lambda_k^+ \mapsto +\infty,$$

$$0 > \lambda_1^- \ge \lambda_2^- \ge \ \ldots \ \ge \lambda_k^- \mapsto -\infty,$$

and has no other eigenvalues. In this case λ_1^+ *and* λ_1^- *are simple principal eigenvalues of this problem, i.e. the corresponding eigenfunctions* $u_1^+(x)$ *and* $u_1^-(x)$ *have no zeros in the interval* I.

Remark 1. We have the following relations:

$$\int_0^1 r\,(u_k^+)^2\,dx > 0 \;\text{ и }\; \int_0^1 r\,(u_k^-)^2\,dx < 0,\;\; k \in \mathrm{N}. \qquad (7)$$

Chapter II consisting of 6 Sections is devoted to the study of the global bifurcation of solutions from zero and from infinity of nonlinear linearizable Sturmian systems with an indefinite weight.

Section 2.1 provides a statement of the problem and provides historical remarks.

Consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(u) = \lambda\, r(x)\, u + g(x,u,u',u'',u''',\lambda),\; 0 < x < 1, \\ u \in BC, \end{cases} \qquad (8)$$

where BC is the set of functions satisfying boundary conditions (6), while the coefficients in the equation and boundary conditions satisfy the condition imposed on them in Chapter I. Moreover, the real-valued function $g \in C([0,\,1] \times R^5)$ satisfies the condition:

$$g\,(x,u,s,\vartheta,w,\lambda) = o(|\,u\,|+|\,s\,|+|\,\vartheta\,|+|\,w\,|) \;\text{ as }\; |\,u\,|+|\,s\,|+|\,\vartheta\,|+|\,w| \to 0, \qquad (9)$$

or

$$g(x,u,s,\vartheta,w,\lambda) = o(|\,u\,|+|\,s\,|+|\,\vartheta\,|+|\,w\,|) \;\text{ as }\; |\,u\,|+|\,s\,|+|\,\vartheta\,|+|\,w| \to \infty, \qquad (10)$$

uniformly on $x \in [0,\,1]$ and $\lambda \in \Lambda$ for each bounded interval $\Lambda \subset R.$

Global bifurcation of solutions of problem (8) for $r(x) > 0,\; x \in [0,\,1],$ was studied in detail in the papers of Z.S.Aliyev[2] and Z.S.Aliyev and. N.A.Mustafayeva[9] subject to the conditions (9) and (10), respectively.

Section 2.2 provides the necessary information from the theory bifurcation of nonlinear eigenvalue problems.

In 2.3 we give a class of functions $S_k^\nu,\; k \in \mathrm{N},\; \nu \in \{+,-\},$ of Banach space $E = C^3[0,\,1] \cap BC$ with the usual norm

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[9] Aliyev, Z. S., Mustafayeva, N.A. Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth order ordinary differential equations // Elec. J. Diff. Equat., – 2018. v. 2018, no. 98, – p. 1–19

$$\| u \|_3 = \| u \|_\infty + \| u' \|_\infty + \| u'' \|_\infty + \| u''' \|_\infty ,$$

where

$$|| u ||_\infty = \max_{x \in [0,\ l]} | u(x) | .$$

It follows from Theorem 1.1 of the paper by Z.S.Aliyev[2] and Theorem 3 that $u_1^+ , u_1^- \in S_1$.

Remark 2. Without loss of generality we can assume that $u_1^+ , u_1^- \in S_1^+$ и $\| u_1^\sigma \|_3 = 1,\ \sigma \in \{+ , -\}$.

In 2.4 we study global bifurcation of nontrivial solutions from zero of nonlinear eigenvalue problem (8).

Denote by C the set of solutions of problem (8).

The following theorem is the main result of this section.

Theorem 4. *Let condition (8) be satisfied. Then for each* $\sigma \in \{ + , - \}$ *and each* $\nu \in \{ + , - \}$ *there exists a continuum* $C_1^{\sigma,\nu}$ *of solutions to problem (8), which contains* $(\lambda_1^\sigma ,\ 0)$, *is contained in* $(R \times S_1^\nu) \bigcup \{(\lambda_1^\sigma ,\ 0)\}$ *and is bounded in* $R \times E$.

In Section 2.5, classes of functions $S_k^{\nu,\sigma},\ k \in \mathrm{N},\ \nu, \sigma \in \{+,-\}$, are constructed and the structures of continua of solutions to problem (8) under an additional condition were studied.

Let the following condition be satisfied

$$u\, g(x,u,s,\vartheta,w,\lambda) \le 0,\ \ (x,u,s,\vartheta,w,\lambda) \in [0,1] \times R^5. \qquad (11)$$

Along with the classes $S_k^\nu,\ k \in \mathrm{N},\ \nu \in \{+,-\}$, consider the classes $S_k^{\sigma,\nu},\ k \in \mathrm{N},\ \sigma, \nu \in \{+,-\}$, which are defined as follows:

$$S_k^{\nu,\sigma} = \{u \in S_k^\nu : \sigma \int_0^1 r u^2 dx > 0\},\ k \in \mathrm{N},\ \sigma, \nu \in \{+,-\}.$$

From relation (7) it follows that $u_1^+ \in S_1^{+,+},\ u_1^- \in S_1^{-,+}$.

Theorem 5. *Let conditions (9) and (11) be satisfied. Then for each* $\sigma \in \{ + , - \}$ *and each* $\nu \in \{ + , - \}$ *the continuum* $C_1^{\sigma,\nu}$ *of solutions of problem (8) is contained in* $(R^\sigma \times S_1^{\sigma,\nu}) \bigcup \{(\lambda_1^\sigma ,\ 0)\}$, *where*

$$R^{+}=(0,\ +\infty),\ R^{-}=(-\infty,0).$$

In 2.6 we study global bifurcation from infinity of the set of solutions of problem (8).

Remark 3. Note that (λ_1^{+},∞) and (λ_1^{-},∞) are asymptotic bifurcation points of problem (8).

Let X_0 be the set of eigenvalues of problem (5), (6).

We have the following global results for problem (8) subject to the condition (10).

Theorem 6. *Let condition (10) be satisfied. Then for each* $\sigma\in\{+,-\}$ *the set* C *contains an unbounded component* $\tilde{C}_1^{\sigma}$, *that intersects the point* $(\lambda_1^{\sigma},\infty)$. *Moreover, if* $\Lambda\subset R$ *is an interval such that* $\Lambda\bigcap X_0=\{\lambda_1^{\sigma}\}$ *and* M_1^{σ} *is a neighborhood of the point* $(\lambda_1^{\sigma},\infty)$ *whose projection on* R *lies in* Λ *and whose projection on* E *is bounded away form* $0\in E$, *then eirher*

1^0. $\tilde{C}_1^{\sigma}\setminus M_1^{\sigma}$ *is bounded in* $R\times E$, *and in this case* $\tilde{C}_1^{\sigma}\setminus M_1^{\sigma}$ *intersects the set* $\{(\lambda,0):\lambda\in R\}$, *or*

2^0. $\tilde{C}_1^{\sigma}\setminus M_1^{\sigma}$ *is unbounded in* $R\times E$, *and if in this case* $\tilde{C}_1^{\sigma}\setminus M_1^{\sigma}$ *has a bounded projection on* R, *then* $\tilde{C}_1^{\sigma}\setminus M_1^{\sigma}$ *intersects the point* $(\lambda_{k'}^{\sigma'},\infty)$, *where* $\lambda_{k'}^{\sigma'}\in X_0$ *and* $(k',\sigma')\neq(1,\sigma)$.

Theorem 7. *For each* $\sigma\in\{+,-\}$ *the component* $\tilde{C}_1^{\sigma}$ *can be decomposed into two subcomponents* $\tilde{C}_1^{\sigma,+}$, $\tilde{C}_1^{\sigma,-}$ *and there exists a neighborhood* $Q_1^{\sigma}\subset M_1^{\sigma}$ *of the point* $(\lambda_k^{\sigma},\infty)$ *such that*

$$((\tilde{C}_1^{\sigma,+}\bigcap Q_1^{\sigma})\setminus\{(\lambda_1^{\sigma},\infty)\})\subset R\times S_1^{+},$$

$$((\tilde{C}_1^{\sigma,-}\bigcap Q_1^{\sigma})\setminus\{(\lambda_1^{\sigma},\infty)\})\subset R\times S_1^{-}.$$

Theorem 8. *Let conditions (10) and (11) be satisfied. Then the following relations hold:*

$$((\tilde{C}_1^{\sigma,+}\bigcap Q_1^{\sigma})\setminus\{(\lambda_1^{\sigma},\infty)\})\subset \hat{R}^{\sigma}\times S_1^{\sigma,+},$$

$$((\tilde{C}_1^{\sigma,-}\bigcap Q_1^{\sigma})\setminus\{(\lambda_1^{\sigma},\infty)\})\subset \hat{R}^{\sigma}\times S_1^{\sigma,-}.$$

In chapter III consisting of four sections we consider nonlinearizable problems for a Sturmian system with an indefinite weight function. We study the structure and behavior of connected components of the set of solutions, bifurcating from the line of trivial solutions and from infinity and contained in the classes of positive and negative functions.

In 3.1 we give a formulation of the proble<m, which considers a nonlinear eigenvalue problem for a completely regular Sturmian system with an indefinite weight.

Consider the following fourth order nonlinear eigenvalue problem

$$\begin{cases} \ell(u) = \lambda r(x)u + g(x,u,u',u'',u''',\lambda),\ 0 < x < 1, \\ u \in BC, \end{cases} \tag{12}$$

where the coefficients in the equation and boundary conditions satisfy the conditions imposed on them in chapter I. Moreover, we represent the nonlinear term h in the form $h = f + g$, where the function $g \in C([0,1] \times R^5; R)$ satisfies the condition (11) and either condition (9) or condition (10), while the function $f \in C([0,1] \times R^5; R)$ satisfies the conditions:

$$uf(x,u,s,\vartheta,\lambda) \le 0,\ (x,u,s,\vartheta,w,\lambda) \in [0,1] \times R^5; \tag{13}$$

there exists a constant $M > 0$ such that

$$\left| \frac{f(x,u,s,\vartheta,w,\lambda)}{u} \right| \le M,\ \ (x,u,s,\vartheta,w,\lambda) \in [0,1] \times R^5. \tag{14}$$

For $r > 0$ and subject to the conditions (9) and (14) the problem (12) was considered in Z.S.Aliyev[2].

In 3.2 we have found the estimation between the principal eigenvalues of the main and perturbed linear problems. Along with problem (5), (6) we consider the spectral problem

$$\begin{cases} \ell(u) + \varphi(x)u = \lambda\, r(x)u,\ 0 < x < 1, \\ u \in BC, \end{cases} \tag{15}$$

where $\varphi(x) \in C[0,1]$ and $\varphi(x) \ge 0, x \in [0,1]$.

Lemma 2. *For each $\sigma \in \{+,-\}$ the following relation holds:*

$$|\tilde{\lambda}_1^\sigma - \lambda_1^\sigma| \leq \frac{\sigma \tilde{M} \int_0^1 (u_1^\sigma(x))^2\, dx}{\int_0^1 r(x)(u_1^\sigma(x))^2\, dx}, \tag{16}$$

where $\tilde{\lambda}_1^\sigma$, $\sigma \in \{+,-\}$, *is a principal eigenvalue of problem (15).*

In 3.3 we study the structure of global continua emanating from intervals of the line of trivial solutions to problem (12) subject to conditions (9) and (14).

Lemma 3. *For each* $\nu \in \{+,-\}$ *and for each small* $\zeta > 0$ *problem (12) has a solution* $(\lambda_\zeta^\nu, u_\zeta^\nu)$ *such that* $u_\zeta^\nu \in S_1^\nu$ *and* $\|u_\zeta^\nu\|_3 = \zeta$.

Corollary 3. *For each* $\nu \in \{+,-\}$ *the set of bifurcation points of problem (12) with respect to the set* $R \times S_1^\nu$ *is non-empty.*

Lemma 4. *If* $(\lambda, 0)$ *is a bifurcation point of problem (12) with respect to the set* $R \times S_1^\nu$, *then*

$$\lambda \in J_1^+ \bigcup J_1^-,$$

where

$$J_1^+ = [\lambda_1^+, \lambda_1^+ + d_1^+],$$

$$J_1^- = [\lambda_1^- - d_1^-, \lambda_1^-],$$

$$d_1^\sigma = \frac{\sigma \tilde{M} \int_0^1 (u_1^\sigma(x))^2\, dx}{\int_0^1 r(x)(u_1^\sigma(x))^2\, dx}, \ \sigma \in \{+,-\}.$$

Denote by D the closure of the set of nontrivial solutions of problem (12).

For each $\sigma \in \{+,-\}$ and each $\nu \in \{+,-\}$ by $\hat{D}_1^{\sigma,\nu}$ we denote the union of all connected components $D_{1,\lambda}^{\sigma,\nu}$ of the set D emanating from the bifurcation points $(\lambda, 0) \in J_1^\sigma \times \{0\}$ with respect to the set $R \times S_1^\nu$. Denote:

$$\tilde{D}_1^{\sigma,\nu} = \hat{D}_1^{\sigma,\nu} \bigcup (J_1^\sigma \times \{0\}).$$

The following theorem is one of the main results

Theorem 9. *For each* $\sigma \in \{+,-\}$ *and each* $\nu \in \{+,-\}$ *the connected component* $\tilde{D}_1^{\sigma,\nu}$ *of the set* D, *containing* $J_1^\sigma \times \{0\}$, *is contained in* $(R \times S_1^\nu) \bigcup (J_1^\sigma \times \{0\})$ *and is unbounded in* $R \times E$.

Theorem 10. *Let* $g \equiv 0$. *Then for each* $\sigma \in \{+,-\}$ *and each* $\nu \in \{+,-\}$ *the connected component* $\tilde{D}_1^{\sigma,\nu}$ *of the set* D, *containing* $J_1^\sigma \times \{0\}$ *is contained in* $(J_1^\sigma \times S_1^\nu) \bigcup (J_1^\sigma \times \{0\})$ *and is unbounded in* $R \times E$.

Let along with conditions (9) and (14) the conditions (11) and (13) also be satisfied. Then the following statements are valid.

Lemma 5. *If* $(\lambda, u) \in D_1^{\sigma,\nu}$, $\sigma \in \{+,-\}$, $\nu \in \{+,-\}$, *then* $\lambda \in R^\sigma$.

Lemma 6. *For each* $\sigma \in \{+,-\}$, *each* $\nu \in \{+,-\}$ *and for each small* $\zeta > 0$ *problem (12) has a solution* $(\lambda_\zeta^{\sigma,\nu}, u_\zeta^{\sigma,\nu})$ *such that* $u_\zeta^{\sigma,\nu} \in S_1^{\sigma,\nu}$ *and* $\| u_\zeta^{\sigma,\nu} \|_3 = \zeta$.

Corollary 4. *For each* $\sigma \in \{+,-\}$ *and each* $\nu \in \{+,-\}$ *the set of bifurcation points of problem (12) with respect to* $R \times S_1^{\sigma,\nu}$ *is non-empty.*

Lemma 7. *If* $(\lambda, 0)$ *is a bifurcation point of (12) with respect to the set* $R \times S_1^{\sigma,\nu}$, *then* $\lambda \in J_1^\sigma$.

Then following theorem is one of the main results of this chapter

Theorem 11. *For each* $\sigma \in \{+,-\}$ *and each* $\nu \in \{+,-\}$ *the component* $\tilde{D}_1^{\sigma,\nu}$ *of the set* D, *containing* $J_1^\sigma \times \{0\}$, *is contained in* $(\tilde{R}^\sigma \times S_1^{\sigma,\nu}) \bigcup (J_1^\sigma \times \{0\})$ *and is unbounded in* $R \times E$.

In 3.4 we study global bifurcation of solution from infinity of nonlinear problem (12) both subject to the condition (10) and (14), and also subject to the condition (10), (11) and (13), (14).

Let conditions (10) and (14) be satisfied. Then the following result holds.

Lemma 8. *For each* $\sigma \in \{+,-\}$, each $\nu \in \{+,-\}$ *and for each sufficiently large* $R_1 > 0$ *the problem (12) has a solution* $(\lambda_{R_1}^{\sigma,\nu}, u_{R_1}^{\sigma,\nu})$ *such that*

$$u_{R_1}^{\sigma,\nu} \in S_1^{\nu} \;u\; \| u_{R_1}^{\sigma,\nu} \|_3 = R_1.$$

Corollary 5. *The set of asymptotic bifurcation point of problem (12) with respect to the set* $R \times S_1^{\nu}$ *is non-empty. Moreover, if* (λ, ∞) *is a asymptotic bifurcation point for (12) with respect to the set* $R \times S_1^{\nu}$, *then* $\lambda \in J_1^{+} \bigcup J_1^{-}$.

For each $\sigma \in \{+,-\}$ and each $\nu \in \{+,-\}$ we determine the set $\overline{D}_1^{\sigma,\nu}$ as a union of all components of the set D, that meet $J_1^{\sigma} \times \{\infty\}$ through $R \times S_1^{\nu}$, and let

$$D_1^{\sigma,\nu} = \overline{D}_1^{\sigma,\nu} \bigcup (J_1^{\sigma} \times \{\infty\}).$$

The following theorem is the main theorem of this section.

Theorem 12. *Let*

$$P_1^{\sigma} = \{(\lambda, u) \in R \times E : dist\{\lambda, J_1^{\sigma}\} \le \delta_1, \| u \|_3 > R_1\}, \; \sigma \in \{+,-\}.$$

Then either

1⁰) $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *is bounded in* $R \times E$, *in this case* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *intersects the set* $\{(\lambda, 0) : \lambda \in R\}$, *or*

2⁰) $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *is unbounded in* $R \times E$, *in this case if* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *has a bounded projection in* R, *then* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *intersects* $J_{k'}^{\sigma'} \times \{\infty\}$ *for some* $(k', \sigma') \ne (1, \sigma)$, *where* $J_{k'}^{\sigma'} \times \{\infty\}$ *is a bifurcation interval of problem (12) such that* $\lambda_{k'}^{\sigma'} \in J_{k'}^{\sigma'}$.

If in addition condition (11) and (13) are satisfied, then the following statement holds.

Lemma 9. *For each* $\sigma \in \{+,-\}$, *each* $\nu \in \{+,-\}$ *and every sufficiently large* $\tilde{R}_1 > 0$ *problem (12) has a solution* $(\lambda_{\tilde{R}_1}^{\sigma,\nu}, u_{\tilde{R}_1}^{\sigma,\nu})$ *such that*

$$u_{\tilde{R}_1}^{\sigma,\nu} \in S_1^{\sigma,\nu} \text{ and } \| u_{\tilde{R}_1}^{\sigma,\nu} \|_3 = \tilde{R}_1.$$

Corollary 6. *The set of asymptotic bifurcation points of problem (12) with respect to the set* $R \times S_1^{\sigma,\nu}$ *is non-empty. Moreover, if* (λ, ∞) *is a bifurcation point for (12) with respect to the set* $R \times S_1^{\sigma,\nu}$*, then* $\lambda \in J_1^{\sigma}$.

Theorem 13. *One of the following statements holds:*

1^0*)* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *is bounded in* $R \times E$*, and in this case* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *meets the set* $\{(\lambda, 0) : \lambda \in R\}$;

2^0*)* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *is unbounded in* $R \times E$*, and if in this case* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *has a bounded projection on* R*, then* $D_1^{\sigma,\nu} \setminus P_1^{\sigma}$ *intersects* $J_k^{\sigma} \times \{\infty\}$ *for some* $k \neq 1$*, where* $J_k^{\sigma} \times \{\infty\}$ *is a bifurcation interval of problem (12) such that* $\lambda_k^{\sigma} \in J_k^{\sigma}$.

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CONCLUSION

In the dissertation work, we study oscillation properties of eigenfunctions of linear problems and study bifurcation of solutions from zero and infinity of nonlinear eigenvalue problems for fourth-order ordinary differential equations (more exactly, for completely regular fourth-order Sturmian systems) with an indefinite weight function.

The following main results were obtained:

– the existence of two infinitely increasing and infinitely decreasing sequences of positive and negative simple eigenvalues of a linear problem for a fourth-order Sturmian systems with an indefinite weight function was proved;

– oscillation properties of eigenfunctions corresponding both to positive and negative first eigenvalues of a fourth-order Sturmian systems with an indefinite weight function were studied;

– the existence of four unbounded continua of solutions bifurcating from points of the line of trivial solutions and $R \times \{\infty\}$, and contained in the classes of functions possessing oscillation properties of first eigenfunctions of corresponding linear problems was shown;

– the existence of four unbounded continua of solutions bifurcating from intervals of the line of trivial solutions and $R \times \{\infty\}$, and contained in the classes of positive and negative functions was proved.

Main results of the dissertation were published in the following papers:

1. Huseynova, R.A. Global bifurcation of solutions of the half-linearizable eigenvalue problems of fourth order // Abstracts of Azerbaijan-Turkey-Ukrainian International Conference MADEA-7 of Mathematical Analysis, Differential Equations and their Applications, – Baku: 08–13 September, – 2015, – p. 71–72.
2. Huseynova, R.A. Global bifurcation from principal eigenvalues for some Sturmian system with sign-changing weight // International Workshop on "Non-harmonic Analysis and Differential Operators", – Baku: –26-27 May, – 2016, – p.50
3. Aliyev, Z.S. Huseynova, R.A. On fourth-order eigenvalue problems with indefinite weight // Trans. of Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, – 2016. v. 36, №4, – p. 37–46.
4. Huseynova, R.A. Global bifurcation from principal eigenvalues for nonlinear fourth order eigenvalue problem with indefinite weight // Proc. of Inst. Math. Mech. Natl. Acad. Sci. Azerb., – 2016. v. 42, №2, – p. 202–211.
5.Aliyev, Z.S. Huseynova, R.A. Bifurcation in nonlinearizable eigenvalue problems for ordinary differential equations of fourth order with indefinite weight // Electron. J. Qual. Theory Differ. Equat., – 2017. № 92, – p. 1–12.
6. Aliyev, Z.S. Huseynova, R.A. Global bifurcation in some nonlinearizable eigenvalue problems with indefinite weight // Abstracts of VIII Annual International Conference of the Georgian Mathematical Union, – Batumi: 4-8 September, – 2017, – p. 51-52.
7. Aliev, Z.S. Huseynova, R.A. Global bifurcation from infinity in some nonlinearizable eigenvalue problems with indefinite weight // Proc. of Inst. Math. Mech. Natl. Acad. Sci. Azerb., – 2018, v. 44, no. 1, – p. 123–134.
8. Huseynova, R.A. Global Bifurcation from zero and infinity in nonlinear beam equation with indefinite weight //- Baku: Caspian J. Appl. Math., Ecol., Econ., –2017, № 2, – p.74–84.

9. Aliev, Z.S. Huseynova, R.A. Global bifurcation from infinity in some fourth order nonlinear eigenvalue problems with indefinite weight // International Conference "Modern Problems of Mathematics and Mechanics " dedicated to the 110th anniversary of Ibrahim Ibrahimov, – Baku: –29 June – 01 July, – 2022, – p. 38–40.

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