REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

SPECTRAL PROPERTIES OF THE SYSTEM OF EIGEN AND ASSOCIATED FUNCTIONS OF THIRD ORDER MATRIX COEFFICIENT DIFFERENTIAL OPERATOR

Specialty: 1211.01 – Differential equation

Field of science: Mathematics

Applicant: **Yuliya Gabil Abbasova**

Baku – 2024

The work was performed at the department of "Functional analysis" of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan

Scientific supervisor: doctor of mathematical sciences, professor
Vali Maharram Kurbanov

Official opponents:
doctor of physical-mathematical sciences, prof.
Nizameddin Shirin Iskenderov

candidate of physical-mathematical sciences, ass. prof.
Elvin Ibrahim Azizbeyov

candidate of physical-mathematical sciences
Nazim Jafar Jafarov

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Chairman of the Dissertation council:
corr.-member of the ANAS, doctor of phys.-math. sc., prof.
____________ **Misir Jumail Mardanov**

Scientific secretary of the Dissertation council:
candidate of phys.-math. sc.
____________ **Abdurrahim Farman Guliyev**

Chairman of the scientific seminar:
academician, doctor of phys.-math. sc., prof.
____________ **Yusif Abulfat Mamedov**

GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic:

The dissertation work has been devoted to studying some spectral properties of the system of the root (eigen and a associated) functions of third order matrix coeffcient ordinary differential operators.

It is known that spectral theory of ordinary differential equations begins with classic works of Sh. Sturm, C. Liouville and then afterwards with the works of V.A.Steklov, D.Ya.Tamarkin, D.Birkhoof, M.L.Rasulov and other famous mathematicians, where the problems of convergence of asymptotics and spectral expansions of eigen values of various boundary value problems were studied.

The study of the following issues in researching spectral theory of differential operators are the most important ones: the basicity of the system of eigen and associated functions of the studied differential operator in these or other spaces; absolute and uniform convergence of spectral expansion of functions included or not included into the domain of definition of the differential operator; equiconvergence of spectral expansion of functions from this or other spaces by the eigen and associated functions of the differential operator with trigonometric Fourier series of this function, etc..

It is clear from the study of not self-associated boundary value problems that the system of eigen functions of such operators, in general, not only does not form a basicity in L_2 and it may not be complete in the L_2 class. Therefore, such systems must be completed with associated functions. In these problems the system of eigen and associated functions (root functions) in general, is not orthogonal in the L_2 space and neither its closeness nor minimality does not provide its basicity in this space. So, the study of not self-associated problems requires new approaches. In this direction, the completeness of the system of eigen and associated functions in L_2 structured specially for a wide class of boundary value problems was proved by M.V.Keldysh.

The study of the completeness for a wide class of boundary value problems was continued by many mathematicions. The basicity of the system of eigen and associated functions of strong regular boundary value problems in L_2 has been shown by V.P.Mikhailov and G.M.Keselman. The block (basicity or parenthetical) basicity of the system of eigen and associated functions of regular problems was indicated in A.A.Shkalikov's work.

The first important results for regular eguiconvergence of regular boundary condition ordinary differential equations were obtained by D.Ya.Tamarkin. Afterwards, the similar result for summable coefficient differential operators was obtained by M.Stone. D.Ya.Tamarkin's equiconvergence theorem was generalized by A.P.Khromov for integral operators generalizing the properties of the Green's function of a differential operator with regular boundary conditions kernel. The above results are based on the resolvent method, and equiconvergence obtained in these works is a block eguiconvergence (parenthetical equiconvergence).

Another method for studying spectral properties of differential operators was suggested by academician V.A.Il'in. V.A.Il'in has clarifed that when the total sum of associated functions is infinite, unlike completeness properties of the system of eigen and associated functions, the basicity and equiconvergence properties are strongly dependent on the choice of associated functions, and are not only determined by the special form of the boundary condition.

These properties are strongly influenced by the values of the coefficients of the differential operator, i.e. a negligible change by retaining these coefficients in its class can cause this property to appear or to disappear. In this condition the basicity and equiconvergence conditions can not be expressed by the terms of boundary conditions. For this reason, V.A.Il'in suggested to determine the eigen and associated functions of a differential operator as a regular solution of a spectral parameter differnetial equation without bounding with specific boundary conditions. This approach enables to consider any boundary conditions (both local and non-local), the system of functions not bounded by any boundary condition and also the systems obtained from the combination of subsets of the systems of eigen and associated functions of two various boundary value problems.

In his works V.A. Il'in has considered a systen of eigen and associated functions of an ordinary differential operator and proved regular convergence and basicity, unconditional basicity theorems in a compact within certain natural conditions.

These studies were continued in V.A.Il'in followers in different directions. These issues were studied in the works of V.V.Tikhomirov, Sh.A.Alimov, I.Yo, I.S.Lomov, V.B.Kerimov, V.D.Budayev, V.I.Komornik, N.Lajetic, V.M.Kurbanov, L.V.Krichkov and others.

Recent years, the dependence of the rate of convergence and equiconvergence on different characteristics is studied intersively and in this direction important results have been obtained in the works of V.M.Kurbanov and A.T.Garayeva, V.M.Kurbanov and R.A.Safarov, I.S.Lomov, A.S.Markov. These problems for second and fourth order matrix coefficient differential equations were studied in the works of V.M.Kurbanov and A.T.Garayeva, A.T.Garayeva, V.M.Kurbanov and Y.I.Huseynova.

Thus, the study of these or other issues for odd order differential equations, in the special case for a third order matrix coefficient operator by the Il'in method is of great mathemtical interest.

Object and subject of the study. Studying regular convergence of spectral expansion by the root functions of a third order matrix coefficient differential operator and componentwise regular equiconvergence rate.

Research goals and objectives. Investigation of absolute and regular summation of spectral expansions the root functions of the differential operator with matrix coefficients of the third order and the speed of regular equiconvergence on the component.

Research methods. In the dissertation work the methods of spectral theory of differential operators, functional analysis and harmonic analysis were used.

The main theses to be defended.

- The results of absolute convergence and regular convergence rate of spectral expansion by the eigen functions of third arder summable matrix coefficient ordinary differetial operator.
- The results of absolute and uniform convergence of biorthogonal expansion of vector-functions from the class $W_{2,m}^{1}(G)$ by the root vector-functions of a third order smooth matrix coefficient differential operator.
- The results of study of influence of biorthogonal expansion of integral continuity modulus of the coefficient of the first order derivative on componentwise regular equicionvergence rate on a compact of ordinary trigonometric Fourier series.
- The result of the study of componentwise equiconvergence rate of vector-functions on a compact from Sobolev, Nikolsky, Besov functional spaces.

Scientific novelty of the research. The following main results were obtained:

- Absolute and regular convergence of a vector-function from the class $f(x) \in W_{1,m}^{1}(G)$, $G = (0,1)$, by the eigen vector-functions of a third order matrix coefficient differental operator was studied and the residue of this expansion was estimated in the metrics $C(\overline{G})$.
- Absolute and regular convergence of a vector-function from the class $W_{p,m}^{1}(G)$, $p > 1$, by the eigen vector-functions of a third order matrix coefficient differental operator was studied, sufficient conditions for the expansion were found and regular convergence conditions on the segment $\overline{G} = [0,1]$ were estimated.
- Theorems on regular componentwise equiconvergence on a compact of spectral expansion of a vector-function from the class $L_p^m(G), p \geq 1$, by the root vector-functions of third order summable matrix coefficient differential operator were proved.

Equiconvergence rate for the functions from the functional spaces $H^{\omega}_{p,m}(G)$, $B^{\alpha}_{p,\theta,m}(G)$, $W^{1}_{1,m}(G)$ was estimated.

- A theorem on absolute and regular convergence of biorthogonal expansion of a vector-function from the class $W^{1}_{2,m}(G)$, $G=(0,1)$, by the root vector-functions of a third order smooth matrix coefficient differential operator was proved and regular convergence rate on the segment $\overline{G}=[0,1]$ was estimated.

Theoretical and practical importance of the research.

The results obtained in the dissertation work are of theoretical character. The obtained results can be used in spectral theory of differential operators, in substantiating the Fourier method when solving mathematical physics problems and in theory of approximation of functions

Approbation and application. The main results of the work were reported in the VII Azerbaijan-Turkey-Ukraine International conference MADEA-7 (Baku, 2015), in the Republican conference "Functional analysis and its applications" devoted to 100-th anniversary of the honored scientist prof. A.Sh.Habibzade (Baku, 2016), in the International conference "Theoretical and applied problems of mathematics" (Sumgayit 2017), in the International conference "Mathematical advances and applications" ICOMAA (Istanbul, Turkey 2018), in the International conference devoted to the 60-th anniversary of IMM of ANAS (Baku 2019), in the Scientific seminar of the department of "Functional analysis" of IMM (prof. H.I.Aslanov), in the Proceedings of the International conference "Modern problems of theory of boundary value problems' of Voronej Spring Mathematical School, Pontryagin readings XXXIII devoted to Y.I.Sapronov's 75 years, (Voronej 2022), "Modern problems of mathematics and mechanics" Proceedings of the International scientific conference devoted to the 110-the anniversary of academician Ibrahim Ibrahimov (Baku, 2022).

Author's personal contribution. The obtained results and suggestions belong to the author.

Author's publications. Complete topic of the author was published in her 12 scientific papers the list of publications is at the end of the thesis.

The name of the organization where the work was executed. The dissertation work has been executed in the department of "Functional analysis" of Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

The total volume of the dissertation work indicating separately the volume of each structural unit. The total volume of the dissertation work–187922 signs (title page–378 signs, contents–2000 signs, introduction–43544 signs, chapter I –76000 signs, chapter II –64000 signs, conclusion–2000 signs). The list of the used references consists of 81 titles.

THE CONTENT OF THE DİSSERTATİON

The dissertation work consists of introduction, 2 chapters, conclusions and a list of references.

Each chapter was separated into sections.

In chapter I of the work, on the interval $G=(0,1)$ the absolute and regular convergence of spectral expansion of absolutely continuous vector-functions by eigen-functions of a matrix coefficient differential operator

$$L\psi=\psi^{(3)}+U_1(x)\psi^{(2)}+U_2(x)\psi^{(1)}+U_3(x)\psi$$

in the interval $G=(0,1)$ is studied.

Sufficient conditions for absolute and regular convergence of orthogonal expansion of a vector-function from the class $W_{p,m}^1(G), p\geq 1,$ is obtained in the segment $\bar{G}=[0,1]$, and regular convergence conditions are estimated.

In section **1.1** we consider the differential operator L**,** study absolute and regular convergence of the expansion of absolutely continuous vector-function by the vector-functions of the considered operator in the case, $U_1(x) \equiv 0$, and estimate the residue of this expansion.

Let us consider on the interval $G = (0,1)$ the operator

$$L\psi = \psi^{(3)} + U_2(x)\psi^{(1)} + U_3(x)\psi$$

Here $U_\ell(x) = \left(u_{\ell ij}(x)\right)_{i,j=1}^{m}$, $\ell = 2,3$, $u_{\ell ij}(x) \in L_1(G)$.

In the segment $\bar{G} = [0,1]$ by $D(G)$ we denote the class of m-component vector-functions with absolutely continuous second order derivatives $\left(D(G) = W_{1,m}^3(G)\right)$.

Under the eigen-function corresponding to the eigen-value λ of the operator L we understand any identically non-zero function $\psi(x) = \left(\psi_1(x), \psi_2(x), \ldots, \psi_m(x)\right)^T \in D(G)$, satisfying almost everywhere the equation $L\psi(x) + \lambda\psi = 0$ on G.

Assume that $L_p^m(G)$, $p \ge 1$ is an m component space of vector-functions $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$. The norm in this space is determined by the equality

$$\|f\|_{p,m} = \left\{ \int_G |f(x)|^p dx \right\}^{1/p} = \left\{ \int_G \left(\sum_{\ell=1}^{m} |f_\ell(x)|^2 \right)^{p/2} dx \right\}^{1/p},$$

in the space case for $p = \infty$,

$$\|f\|_{\infty,m} = vrai \sup_{x \in \bar{G}} |f(x)|.$$

If the function $f(x)$ is absolutely continuous in $\bar{G}$ and $f'(x) \in L_p^m(G)$, we say that the function $f(x)$ is contained in the class $W_{p,m}^1(G)$, $p \ge 1$. In $W_{p,m}^1(G)$ the norm is determined by the following equality

$$\|f\|_{W_{p,m}^1(G)} = \|f\|_{p,m} + \|f'\|_{p,m}.$$

Assume that $\{\psi_k(x)\}_{k=1}^{\infty}$ is a complete orthonormal system from the space $L_2^m(G)$ consisting of the eigen-functions of the operator L.

$\{\lambda_k\}_{k=1}^{\infty}$ is an appropriate system of eigen values $(Re\,\lambda_k = 0)$. Let us introduce the spectral parameter μ_k:

$$\mu_k = \begin{cases} (-i\lambda_k)^{1/3}, & \operatorname{Im}\lambda_k \ge 0, \\ (i\lambda_k)^{1/3}, & \operatorname{Im}\lambda_k < 0. \end{cases}$$

Introduce the special sum of the orthogonal expansion of the function $f(x) \in W_{1,m}^1(G)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x,f) = \sum_{\mu_k \le \nu} f_k \psi_k(x),\ \nu > 0,$$

here we introduce the difference

$$f_k = (f, \psi_k) = \int_0^1 < f(x), \psi_k(x) > dx,$$

$$< f(x), \psi_k(x) > = \sum_{\ell=1}^{m} f_\ell(x) \overline{\psi_{k\ell}(x)},$$

$$\psi_k(x) = (\psi_{k1}(x), \psi_{k2}(x), \ldots, \psi_{km}(x))^T.\ R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$$

The main results of this section are in the following theorem.

Theorem 1. Assume that $f(x) \in W_{1,m}^1(G)$ the system $\{\psi_k(x)\}_{k=1}^{\infty}$ is regularly bounded and the conditions

$$\left| < f(x), \psi_k^{(2)}(x) > \Big|_0^1 \right| \le C(f)\mu_k^\alpha,\ 0 \le \alpha < 2, \mu_k \ge 8\pi, \tag{1}$$

$$\sum_{n=2}^{\infty} n^{-1} \omega_{1,m}\left(f', n^{-1}\right) < \infty \tag{2}$$

are satisfied. Then the expansion of the vector function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and regularly in the interval $\overline{G} = [0,1]$ and the estimation

$$\|R_\nu(\cdot,\ f)\|_{C[0,1]} \le const\Big\{\ C(f)\nu^{\alpha-2} + \sum_{n=[\nu]}^{\infty} n^{-1}\omega_{1,m}\left(f', n^{-1}\right)$$

$$+\left(\|f\|_{\infty,m} + \|f'\|_{1,m}\right)\nu^{-1}\sum_{r=2}^{3}\|U_r\|_1 \nu^{2-r} + \nu^{-1}\|f'\|_{1,m}\ \Big\}, \qquad (3)$$

is valid. Here $\omega_{1,m}(g,\delta)$ is an integral continuity modulus of the function

$$g(x) = (g_1(x), g_2(x), \dots, g_m(x))^T \in L_1^m(G);$$

$$\left(\omega_{p,m}(f,\delta) \equiv \omega_p(f,\delta) = \sup_{0<h\le\delta}\left\{\int_0^{1-h} |f(x+h) - f(x)|^p dx\right\}^{1/p}\right).$$

$\|U_r\|_1 = \sum_{i,j=1}^{m}\|u_{rij}\|_1$, $r = 2,3;$ $const$ is independent of the function $f(x)$.

We get the following corollaries from theorem 1.

Corollary 1. If the system $\{\psi_k(x)\}_{k=1}^{\infty}$ is regularly bounded, then
$f(x) \in W_{1,m}^1(G)$, $f(0) = f(1) = 0$ and $f'(x) \in H_{1,m}^{\alpha}(G), 0 < \alpha < 1,$ ($H_{1,m}^{\alpha}(G)$ is a Nikolski class of m component vector-functions).

Then the estimation

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \le const\, \nu^{-\alpha}\|f'\|_{1,m}^{\alpha}$$

is valid, here

$$\|g\|_{1,m}^{\alpha} = \|g\|_{1,m} + \sup_{\delta>0}\delta^{-\alpha}\omega_{1,m}(g,\delta)\ .$$

Corollary 2. If the system $\{\psi_k(x)\}$ is regularly bounded, $f(x) \in W_{1,m}^1(G)$, $f(0) = f(1) = 0$ and for certain $\beta > 0$ the estimation
$\omega_{1,m}(f',\delta) = O\left(ln^{-(1+\beta)}\delta^{-1}\right)$, $\delta \to +0,$ is satisfied. Then

$$\|R_\nu(\cdot, f)\|_{C[0,1]} = \mathrm{O}(\ln^{-\beta}\nu),\ \ \nu \to \infty.$$

In section **1.2** we study absolute and regular convergence of orthogonal expansion of the function from the class $W^1_{p,m}(G),\ 1<p\le\infty$ by the eigen-functions of the operator L.

Theorem 2. Assume that
$U_1(x)\equiv 0, U_r(x)\in L_1(G), r=\overline{2,3};\ \ f(x)\in W^1_{p,m}(G),\ \ p>1$, and the condition

$$\left| < f(x), \psi_k^{(2)}(x) > \Big|_0^1 \right| \le C_1(f)\mu_k^{\alpha}\left\|\psi_k\right\|_{\infty,m},\ 0\le\alpha<2,\ \mu_k\ge 1, \quad (4)$$

is satisfied. Here the constant $C_1(f)$ is dependent on the function $f(x)$.

Then spectral expansion of the function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and regularly in the interval $\overline{G}=[0,1]$ and for the residue $R_\nu(x,f)$ the estimation

$$\left\|R_\nu(\cdot,f)\right\|_{C[0,1]}\le const\left\{\ C_1(f)\nu^{\alpha-2}+\nu^{-\beta}\left\|f'\right\|_{p,m}+\right.$$

$$\left.+\nu^{-1}\left(\left\|f\right\|_{\infty,m}+\left\|f'\right\|_{1,m}\right)\sum_{r=2}^{3}\nu^{2-r}\left\|U_r\right\|_1\ \right\}, \quad (5)$$

is satisfied. Here $\beta=\min\left\{\frac{1}{2},\frac{1}{q}\right\},\ \ p^{-1}+q^{-1}=1;\ \ \nu\ge 2;$

$const$ $f(x)$ is independent of the function $\left\|U_r\right\|_1=\sum_{i,j=1}^{m}\left\|u_{rij}\right\|_1$.

Corollary 3. If in theorem 2 the vector-function $f(x)$ satisfies the relation $f(0)=f(1)=0$ the condition (4) will be a priori satisfied and the estimation

$$\left\|R_\nu(\cdot,f)\right\|_{C[0,1]}\le const\,\nu^{-\beta}\left\|f'\right\|_{p,m},\ \nu\ge 2;$$

is valid. If $C_1(f)=0$ or $0\le\alpha<2-\beta$ then the estimation

$$\left\|R_\nu(\cdot,f)\right\|_{C[0,1]}=o\left(\nu^{-\beta}\right),\ \ \nu\to+\infty\ ,$$

is valid. Here the symbol "o" is dependent on the function $f(x)$.

Theorem 3. Assume that $U_1(x)\in L_2(G), U_r(x)\in L_1(G),\ r=\overline{2,3};\ \ f(x)\in W^1_{2,m}(G)$ and condition (4) is satisfied.

Then the spectral expansion of the vector-function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and regularly in the interval $\overline{G}=[0,1]$, and the estimation

$$\|R_\nu(\cdot,f)\|_{C[0,1]}\le const\left\{\ C_1(f)\nu^{\alpha-2}+\nu^{-\frac{1}{2}}\left(\|U_1^* f\|_{2,m}+\|f'\|_{2,m}\right)+\right.$$

$$\left.+\nu^{-1}\|f\|_{\infty,m}\sum_{r=2}^{3}\nu^{2-r}\|U_r\|_1\ \right\},\ \ \nu\ge 2, \qquad (6)$$

is valid. Here the $const$ is independent of $f(x)$.

Corollary 4. If in theorem 3 $C_1(f)=0$ or the condition $0\le\alpha<3/2$ is satisfied, then the estimation

$$\|R_\nu(\cdot,f)\|_{C[0,1]}=o\left(\nu^{-\frac{1}{2}}\right),\quad \nu\to+\infty,$$

is valid. Here the symbol "o" is dependent on the function $f(x)$.

Theorem 4. Assume that the condition $U_1(x)\in L_2(G), U_r(x)\in L_1(G),\ r=\overline{2,3};\quad f(x)\in W^1_{p,m}(G),\ 1<p<2$ (4) is satisfied and the system $\{\psi_k(x)\}_{k=1}^{\infty}$ is regularly bounded.

Then the spectral expansion of the vector-function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and regularly ın the ınterval $\overline{G}=[0,1]$, the estimation:

$$\|R_\nu(\cdot,f)\|_{C[0,1]}\le const\left\{\ C_1(f)\nu^{\alpha-2}+\nu^{-\frac{1}{2}}\|U_1^* f\|_{2,m}+\right.$$

$$\left.+\nu^{-\frac{1}{q}}\|f'\|_{p,m}+\nu^{-1}\|f\|_{\infty,m}\sum_{r=2}^{3}\nu^{2-r}\|U_r\|_1\ \right\},\ \ \nu\ge 2, \qquad (7)$$

ıs valıd. Here $p^{-1}+q^{-1}=1$, $const$ is independent of $f(x)$.

Corollary 5. İf in theorem 4 $C_1(f)=0$ or the condition $0\le\alpha<2-q^{-1}$ is satisfied, then

$$\|R_\nu(\cdot, f)\|_{C[0,1]}=o\left(\nu^{-\frac{1}{q}}\right), \quad \nu\to+\infty,$$

Here the symbol "o" is dependent on the function $f(x)$.

Note that similar results for the Schrodinger operator were obtained in the papers of N.L.Lajetich, in the case $m=1$ in the works of V.M.Kurbanov and R.A.Safarov, for arbitrary m in the papers of V.M.Kurbanov and A.Garayeva, in the case $m=1$ for second order operators in the papers of V.M.Kurbanov and E.B.Akhundova, for fourth order operators in the works of V.M.Kurbanov and Y.I.Hiuseynova.

In section 1.3 we consider the matrix coefficient operator L $U_\ell(x)=\left(u_{\ell ij}(x)\right)_{i,j=1}^m$, $\ell=\overline{1,3}$, here $u_{\ell ij}(x)\in L_1(G)$, $\ell=2,3$; $u_{1ij}(x)\in L_2(G)$.

We study absolute and regular convergence of spectral expansion of the vector function $f(x)\in W_{1,m}^1(G)$ by the eigen vector functions of the considered operator and estimate regular convergence rate of this expansion.

The main results of this section are in the following theorem

Theorem 5. Assume that $U_1(x)\in L_2(G)$, $U_r(x)\in L_1(G)$, $r=2,3$, the system $f(x)\in W_{1,m}^1(G)$, $\{\psi_k(x)\}_{k=1}^\infty$ is regularly bounded in the interval $\overline{G}=[0,1]$ condition (1) and

$$\sum_{n=2}^{\infty} n^{-1}\omega_{1,m}\left(f', n^{-1}\right)<\infty, \quad \sum_{n=2}^{\infty} n^{-1}\omega_{1,m}\left(U_1^* f, n^{-1}\right)<\infty. \qquad (8)$$

are satisfied.

Then the spectral expansion of the vector-function $f(x)$ by the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and regularly in the interval $\overline{G}=[0,1]$ and the estimation

$$\|R_\nu(\cdot,f)\|_{C[0,1]} \le const\Big\{C(f)\nu^{\alpha-2}+\Big(1+\|U_1\|_1\Big)\Big[\sum_{n=[\nu]}^{\infty} n^{-1}\omega_{1,m}\Big(U_1^* f, n^{-1}\Big)+$$

$$+\sum_{n=[\nu]}^{\infty} n^{-1}\omega_{1,m}\Big(f', n^{-1}\Big)+\nu^{-1}\Big(\|f'\|_{1,m}+\|U_1^* f\|_{1,m}\Big)\Big]+\nu^{-1}\Big(\|f'\|_{1,m}+\|U_1^* f\|_1+$$

$$+\|f\|_{\infty,m}\sum_{r=2}^{3}\nu^{2-r}\|U_r\|_1\Big\},\ \nu\ge 2 \qquad (9)$$

is valid. Here the matrix U_1^* is the conjugation of the matrix U_1; $const$ is independent of the function $f(x)$.

Similar results for second order operators were obtained in the works of A.Garayeva, V.M.Kurbanov and A.Garayeva. Estimation (9) was proved for the operator L in the case $m=1$ in the papers of V.M.Kurbanov and E.B.Akhundov.

Chapter II of the dissertation work considers third order matrix coefficient ordinary differential equations in the interval $G=(0,1)$. Componentwise regular convergence rate of biorthogonal expansion of the considered operator by eigen and associated vector-functions and trigonometric expansion in a compact is studied, absolute and regular convergence of biorthogonal expansion of a vector-function from the class $W_{2,m}^1(G)$, $G=(0,1)$, by the eigen and adjoined vector-functions is researched and regular convergence rate of this expansion in $\overline{G}$ is estimated.

In section **2.1** we consider a matrix coefficient ordinary differential operator

$$L\psi=\psi^{(3)}+U_2(x)\psi^{(1)}+U_3(x)\psi,\ \ (G)=(0,1)$$

here $U_\ell(x)=\big(u_{\ell ij}(x)\big)_{i,j=1}^m$, $\ell=2,3;\ u_{\ell ij}(x)\in L_1(G)$.

Componentwise regular equiconvergence of biorthogonal expansion of the considered operator by eigen and associated vector-functions with trigonometric expansion is studied.

Biorthogonal expansion of the operator L by the root vector-functions and trigonometric expansion of ordinary trigonometric expansion in the compact $K \subset G$ the dependence of regular eguiconvergence rate of continuity modulus of the coefficient $U_2(x)$ on the row elements are studied.

Componentwise regular equiconvergence on a compact for a Schrodinger operator was studied by V.A.Il'in. For arbitrary order differential operator on a compact, componentwise equiconvergence rate in metrics L_p^m, $1 \le p \le \infty$, was studied by V.M.Kurbanov (in the case $1 \le p < \infty$ it was studied in the works of by I.S.Lomov and A.S.Markov, A.S.Markov).

Assume that the function $\omega(t)$ is a non-decreasing continuous function determined in the interval $[0, \infty)$ and satisfies the conditions

1) $\omega(0) = 0, \quad \omega(t) > 0, \ t > 0;$ 2) $\dfrac{\omega(t)}{t}$ is not increasing.

By $H_{p,m}^{\omega}(G), \ p \ge 1$, we denote a set of functions from the space $L_p^m(G)$ and satisfying the condition $\omega_p(f, \delta) \le C(f)\omega(\delta)$, here the function $\omega_p(f, \delta)$ is a continuity modulus of the function $f(x)$ i.e. $L_p^m(G)$

$$\omega_p(f, \delta) \equiv \omega_{p,m}(f, \delta) = \sup_{0<h\le\delta} \left\{ \int_0^{1-h} \left| f(x+h) - f(x) \right|^p dx \right\}^{1/p},$$

$C(f)$ is a constant dependent on $f(x)$. In $H_{p,m}^{\omega}(G)$ the norm is determined by the following equality

$$\| f \|_{p,m}^{\omega} = \| f \|_{p,m} + \sup_{\delta>0} \left(\omega_p(f, \delta) \big/ \omega(\delta) \right).$$

$B_{p,\theta,m}^{\alpha}(G), \ 0 < \alpha < 1, \ 1 \le \theta \le \infty,$ denotes the Besov class. The norm in this space is denoted by the equality

$$\| f \|_{B^{\alpha}_{p,\theta,m}(G)} = \| f \|_{p,m} + \left(\int_0^{h_0} \left(t^{-\alpha-\frac{1}{\theta}} \omega_p(f,t) \right)^{\theta} dt \right)^{\frac{1}{\theta}}, h_o > 0.$$

Note that $B^{\alpha}_{p,\infty,m}(G) = H^{\alpha}_{p,m}(G)$ is a Nikolsky class $(\omega(t) = t^{\alpha})$.

Assume that the system $\{\psi_k(x)\}_{k=1}^{\infty}$,

$\psi_k(x) = (\psi_{k1}(x), \psi_{k2}(x), \ldots, \psi_{k,m}(x))^T$ satisfies the conditions A_p (V.A.Il'in conditions) :

1) for any fixed $p \geq 1$ the system $\{\psi_k(x)\}$ is closed and minimal in $L_p^m(G)$,

2) Each $\psi_k(x)$ component of the vector-function $\psi_k(x)$ and including its derivatives up to the second order are absolutely continuous in the segment $\overline{G}$, each vector-function $\psi_k(x)$ satisfies almost everywhere the equation

$$L\psi_k + \lambda_k \psi_k = \theta_k \psi_{k-1}$$

for a certain complex number λ_k in the interval G, here θ_k equals either 0 (in this case $\psi_k(x)$ is an eigen vector-function), or 1 (in this case $\lambda_k = \lambda_{k-1}$ is not required and $\psi_k(x)$ is said to be an adjoined vector-function) $\theta_1 = 0$;

3) $\mu_k = \begin{cases} (i\lambda_k)^{\frac{1}{3}}, & if \ \operatorname{Im}\lambda_k < 0 \\ (-i\lambda_k)^{\frac{1}{3}}, & if \ \operatorname{Im}\lambda_k \geq 0, \end{cases}$

here the numbers

$(re^{i\varphi})^{1/3} = r^{\frac{1}{3}} e^{i\varphi/3}$, $-\pi < \varphi \leq \pi$, $\rho_k = \operatorname{Re}\mu_k \geq 0$, satisfiey the inequalities

$$|\operatorname{Im}\mu_k| \leq C_1, \quad \forall k \in N \tag{10}$$

$$\sum_{\tau \le \rho_k \le \tau+1} 1 \le C_2, \quad \forall \tau \ge 0 \tag{11}$$

4) For arbitrary $K \subset G$ there exists such a constant $C_0(K)$ that the inequalities

$$\|\psi_k\|_{p,m,K} \|\varphi_k\|_{q,m} \le C_0(K), \quad k = 1,2,\ldots,$$

are satisfied. Here the system $\{\varphi_k\}$ is an orthogonal conjugation of $\{\psi_k\}$ ($\varphi_k \in L_q^m(G)$, $p^{-1} + q^{-1} = 1$; $q = \infty$ if $p = 1$),
$\|\cdot\|_{p,m,K} = \|\cdot\|_{L_p^m(K)}$.

For $p \ge 1$ fixed in conditions A_p we form the ν-th order special sum of biorthogonal expansion of arbitrary function $f(x) \in L_p^m(G)$ by the system $\{\psi_k(x)\}$:

$$\sigma_\nu(x, f) = \sum_{\rho_k \le \nu} (f, \varphi_k)\, \psi_k(x), \quad \nu > 0,$$

here

$$(f, \varphi_k) = f_k = \int_0^1 < f(x), \varphi_k(x) > dx = \int_0^1 \sum_{j=1}^m f_j(x)\, \overline{\varphi_{kj}(x)}\, dx,$$

$\varphi_k(x) = (\varphi_{k1}(x), \varphi_{k2}(x), \ldots, \varphi_{km}(x))^T$.

It is clear that the j-th component of the special sum $\sigma_\nu(x, f)$ is determined by the equality
$\sigma_\nu^j(x, f) = \sum_{\rho_k \le \nu} f_k \psi_{kj}(x)$, $j = \overline{1,m}$, i.e.

$$\sigma_\nu(x, f) = \left(\sigma_\nu^1(x, f), \sigma_\nu^2(x, f), \ldots, \sigma_\nu^m(x, f)\right)^T.$$

By $S_\nu(x, f_j)$ we denote the $\nu-$th order special sum of trigonometric Fourier series of the function $f_j(x)$, $j = \overline{1,m}$,.

We introduce some denotations :

$$\Delta_\nu^j(f, K) = \| \sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j) \|_{C(K)}, \qquad j = \overline{1,m}$$

$$\hat{f}_k = f_k \| \varphi_k \|_{q,m}^{-1} = (f, \varphi_k) \| \varphi_k \|_{q,m}^{-1};$$

$$\psi\left(f, \nu/2, \gamma \right) = \nu^{-1} \sum_{1 \le \rho_k \le \nu/2} \rho_k^{-\gamma} \left| \hat{f}_k \right|, \quad \gamma \ge 0,$$

$$\Phi\left(f, n, \gamma \right) = \sum_{i=1}^{n} i^{-\gamma} \omega_1 \left(f, i^{-1} \right), \quad \gamma \ge 0,$$

$$\Phi_p(f, \nu) = \nu^{-1} \| f \|_{p,m} + \max_{\rho_k \ge \nu/2} \left| \hat{f}_k \right|,$$

$$Q_p(f_j, \nu) = \nu^{-1} \| f_j \|_p + \max_{2\pi k \ge \nu/2} \left| \tilde{f}_{jk} \right|,$$

here the numbers $\tilde{f}_{jk}$ are Fourier coefficients of the function $f_j(x)$ by the trigonometric system normalized in $L_q(G)$;

$$D(\nu, U_2) = \inf_{\substack{\alpha > 1 \\ n \ge 2}} \left\{ \Omega_{1j}\left(U_2, n^{-1} \right) \psi\left(f, \nu/2, 0 \right) + \right.$$

$$\left. + n^{2\left(1-\alpha^{-1}\right)} \| U_2 \|_{1j} \psi\left(f, \nu/2, 1-\alpha^{-1} \right) \right\},$$

here

$$\Omega_{1j}(U_2, \delta) = \max_{1 \le l \le m} \omega_1(u_{2jl}, \delta), \quad \|U_2\|_{rj} = \max_{1 \le l \le m} \|u_{2jl}\|_{L_r(G)};$$

$$T(f, \nu, r) = \psi\left(f, \nu/2, 1 - r^{-1}\right) + \Phi_p(f, \nu),$$

$$T(f, \nu, \infty) = \psi(f, \nu/2, 1) + \Phi_p(f, \nu),$$

$$T_1(f, \nu, r) = \nu^{-1} \left\{ \Phi\left(f, [\nu/2], 1 - r^{-1}\right) + \nu\, \omega_1\left(f, \nu^{-1}\right) + \|f\|_{p,m} \right\};$$

$$\varphi_p(f, \nu) = \omega_1\left(f, \nu^{-1}\right) + \|f\|_{p,m}; \quad \psi_p(f_j, \nu) = \omega_1\left(f_j, \nu^{-1}\right) + \nu^{-1} \|f_j\|_p;$$

$$A(\nu) = \inf_{\substack{\alpha > 1 \\ n \ge 2}} \left\{ \Omega_{1j}\left(U_2, n^{-1}\right) \left(\Phi\left(f, \left[\frac{\nu}{2}\right], 0 \right) + \|f\|_{p,m} \ln \nu \right) + \right.$$

$$\left. + \|U_2\|_{1,j}\, n^{2\left(1-\alpha^{-1}\right)} \left(\Phi\left(f, \left[\frac{\nu}{2}\right], 1 - \alpha^{-1} \right) + \left(1 - \alpha^{-1}\right)^{-1} \|f\|_{p,m} \right) \right\},$$

$$T_2(\nu) = \inf_{n \ge 2} \left\{ \Omega_{1j}\left(U_2, n^{-1}\right) \ln \nu + \|U_2\|_{1,j} \ln n \right\}.$$

Definition 1. If for arbitrary compact $K \subset G$ the condition $\lim_{\nu \to \infty} \Delta_\nu^j\left(f, K\right) = 0$ is satisfied, we say that the j-th component of biorthogonal expansion of the vector-function $f(x)$ by the system

$\{\psi_k(x)\}_{k=1}^{\infty}$ in arbitrary compact $K \subset G$ regularly equiconverge with trigonometric Fourier expansion corresponding to the component $f_j(x)$ of the vector-function $f(x)$.

The main results of this section are in the theorems on componentwise regular equiconvergence.

Theorem 6. Assume that all the elements of the j-th row of the matrix function $U_2(x)$ are contained in the space $L_r(G), r \geq 1;$; $G=(0,1)$ and the system $\{\psi_k(x)\}_{k=1}^{\infty}$ satisfies the conditions A_p for certain fixed $p \geq 1$. Then the j-th components of biorthogonal expansion of arbitrary vector-function $f(x) \in L_p^m(G)$ regularly equiconverge with trigonometric Fourier series corresponding to the component $f_j(x)$ of the vector-function $f(x)$ in arbitrary compact $K \subset G$, and the estimation

$$\Delta_\nu^j(f,K) \leq C(K)\{\|U_2\|_{rj} T(f,\nu,r) + \|U_3\|_{1j} T(f,\nu,\infty) + \\ +\Phi_p(f,\nu) + Q_p(f_j,\nu)\} \qquad for \ \ r>1; \tag{13}$$

$$\Delta_\nu^j(f,K) \leq C(K)\{D(\nu,U_2) + \|U_3\|_{1j} T(f,\nu,\infty) + \\ +\Phi_p(f,\nu) + Q_p(f_j,\nu)\} \qquad for \ \ r=1; \tag{14}$$

are satisfied, here the constant $C(K)$ is independent of the function $f(x)$ and ν .

Theorem 7. Assume that the conditions of theorem 6 are satisfied and the estimations

$$\left|\hat{f}_k\right| \leq const\{\omega_1(f,\rho_k^{-1}) + \rho_k^{-1}\|f\|_{1,m}\}, \ \ \rho_k \geq 1.$$

are satisfied for biorthogonal coefficients f_k of the vector-function $f(x) \in L_p^m(G)$

Then for $r>1$ the estimations

$$\Delta_\nu^j(f,K) \leq C(K)\{\|U_2\|_{rj} T_1(f,\nu,r) + \|U_3\|_{1j} T_1(f,\nu,\infty) + \\ +\varphi_p(f,\nu) + \psi_p(f_j,\nu)\}, \tag{15}$$

for $r=1$ the estimations

$$\Delta_\nu^j(f,K)\le C(K)\left\{\nu^{-1}A(\nu)+\|U_3\|_{1j}T_1(f,\nu,\infty)+\right.$$
$$\left.+\varphi_p(f,\nu)+\psi_p(f_j,\nu)\right\}, \tag{16}$$

are valid, here the constant $C(K)$ is independent of $f(x)$ or ν.

The last section of the dissertation considers a third order complex valued matrix coefficient ordinary differential operator

$$L\psi=\psi^{(3)}+U_1(x)\psi^{(2)}+U_2(x)\psi^{(1)}+U_3(x)\psi$$

Here $U_\ell(x)\in W_1^{3-\ell}(G)$, $\ell=\overline{1,3}$, $u_{\ell ij}(x)\in W_1^{3-\ell}(G)$.

Under certain conditions absolute and regular convergence of biorthogonal expansion of the vector-function $f(x)$ from the class $W_{2,m}^1(G), G=(0,1)$ by the system of eigen and associated vector-functions is studied and regular convergence rate of this expansionin in the segment $\overline{G}$ is estimated.

At the end, I would like to express my deep gratitude to my supervisor V.M.Kurbanov for the problem statement and his valuable advices.

Conclusions

The dissertation work has been devoted to studying some spectral properties of the system of the root (eigen and a associated) functions of third order matrix coeffcient ordinary differential operators.

- Absolute and regular convergence of biorthogonal expansion of vector-function from the class $f(x) \in W_{1,m}^{1}(G),\ G=(0,1),$ by eigen vector-functions of third order matrix coefficient differential operator was proved and the residue of this expansion in the metrics $C(\overline{G})$ was estimated.
- Absolute and regular convergence of spectral expansion of a vector-function from the class $W_{p,m}^{1}(G),\ p>1,$ by the eigen vector-functions of a third order matrix coefficient differential operator was studied, sufficient conditions for this expansion were found and regular convergence rate in the interval $\overline{G}=[0,1]$ was estimated.
- Theorems on componentwise equiconvergence on a compact of a vector-function from the class $L_p^m(G), p \geq 1$, by the root eigen-functions of a third order summable matrix coefficient differential operator with trigonometric series were proved. Componentwise regular equiconvergence rate for functions from the functional spaces $H_{p,m}^{\omega}(G),\ B_{p,\theta,m}^{\alpha}(G),\ W_{1,m}^{1}(G)$ was estimated.
- A theorem on absolute and regular convergence of biorthogonal expansion of a vector-function from the class $W_{2,m}^{1}(G),\ G=(0,1),$ by the root vector-functions of a third order smooth matrix coefficient differential operator was proved and regular convergence rate on the segment $\overline{G}=[0,1]$ was estimated.

The main results of the dissertation work has been published in the following work:

1. Abbasova, Y.G. Absolute convergence of spectral expansion of absolutely continuous vector-function in eigen vector-functions of third order differential operator // Transcations of ANAS, Series of Physical-Tech. and Math. Sciences, -2015. vol.XXXV, № 1, -p. 3-9.
2. Abbasova, Y.G. Absolute and uniform convergence of spectral expansion of absolutely continuous vector-function in eigen vector-functions of third order differential operator // MADEA-7, Azerbaijan-Turkey-Ukrainian Intern. conference. “Mathematical and Differential Equations and the Applications”, -Baku: -08-13 September, -2015, p. 3.
3. Аббасова, Ю.Г. Абсолютная и равномерная сходимость спектрального разложения функции из класса $W_{p,m}^{1}(G)$, $p>1$, по собственным вектор-функциям дифференциального оператора третьего порядка // Əməkdar elm xadimi, prof. Ə.Ş.Həbibzadənin 100-cü ildönümünə həsr olunmuş “Funksional analiz və onun tətbiqləri” adlı respub. elmi konf., -Bakı: BDU, -2016, -s. 53-55.
4. Аббасова, Ю.Г., Исмайлова, А.И. Равномерная сходимость ортогонального разложения абсолютно непрерывной вектор-функции по собственным функциям дифференциального оператора третьего порядка сматричными коэффициентами // -Bakı: Pedaqoji Universitetin Xəbərləri, Riyaziyyat və təbiət elmləri seriyası, -2016. №2, c.64, -s. 9-20.
5. Аббасова, Ю.Г. Покомпонентная расходимость для дифференциального оператора третьего порядка с матричными коэффицентами // Sumqayıt Dövlət Universitetinin yaradılmasının 55 illiyinə həsr olunmuş “Riyaziyyatın nəzəri və tətbiqi problemləri” beynəlxalq elmi konfransı, -Sumqayıt: -25-26 may, -2017, -s.53-54.
6. Курбанов, В.М., Аббасова, Ю.Г.Сходимость спектрального разложения функции из класса $W_{p,m}^{1}(G)$, $p>1$,

по собственным вектор-функциям дифференциального оператора третьего порядка // -Украина: Укр. мат.журн., -2017. т.69, № 6,-с. 719-733.
7. Kurbanov, V.M., Abbasova, Y.G. Convergence of spectral expansion of absolutely continuous vector-function in root vectors of third order differential operator with matrix coefficients // -Baku: Proceedings of Institute Mathamtics and Mechanics of ANAS, -2018. v. 44, № 1, -s. 164-174.
8. Abbasova, Y.G. Componentwise uniform equiconvergence theorem for a third order differential operator// İnternational conference on "Mathematical Advances and Applications, Yıldız Technical University, İCOMAA-2018, İstanbul/Turkey: -11 -13 may, -2018,-p.160
9. Abbasova, Y.G. On the rate of componentvise equiconvergence for a third-order differential operator with matrix coefficients // "Modern Problems of Mathematics and Mechanics" Proceedings of the Intern. confer. devoted to the 60 anniversary of the IMM of ANAS, - 23-25 October, -2019, -s.37-39 .
10. Аббасова, Ю.Г. О скорости покомпонентной равномерной равносходимости для дифференциального оператора третьего порядка с матричными коэффициентами // -Bakı: Pedaqoji Universitetin Xəbərləri, Riyaziyyat və təbiət elmləri seriyası, -2019. c.67, №2, -s. 9-24.
11. Аббасова, Ю.Г. Абсолютная и равномерная сходимость спектрального разложения функции из класса $W_{p,m}^1(G), 1<p<2$ по собственным вектор-функциям дифференциального оператора третьего порядка // "Современные методы теории краевых задач» Матер. межд. конф. Воронежская весенняя матем. школа Понтрягинские Чтения XXXIII посвящ. Ю.И.Сапронову (75- летию со дня рождения) Воронеж Изд. дом ВГУ: -3 -9 мая, -2022, -с. 22-24.
12. Kurbanov, V.M., Abbasova, Yu.G. Convergence of the spectral expansion of a function from the class $W_{p,m}^1(G)$, $1<p<2$, in the vector eigen functions of a differential operator of the third

order // “Modern Problems of Mathematics and Mechanics” Proceedings of the İntern. scientific conf. devoted to the 110–th anniv. of acad. İbrahim İbrahimov, -Baku: -29 June – 1July, -2022, -p.126-127.

The defense will be held on **17 May 2024** year at **14^{00}** at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Address: AZ 1141, Baku city, B.Vahabzade str, 9.

Dissertation is accessible at the library of the Institute of Mathematics and Mechanics of the library.

Electronic versions of the dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics.

Abstract was sent to the required addresses on **15 April 2024**.

Signed for print: 29.03.2024
Paper format: 60x841/16
Volume: 38235
Number of hard copies: 30