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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

ON WELL-POSEDNESS OF SOME INVERSE PROBLEMS FOR PARABOLIC AND HYPERBOLIC TYPE EQUATIONS

Specialty: 1211.01 – Differential equations

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic.

Under the inverse problem we mean the finding of the unknown coefficient or the right hand side involved in the equation when there is additional information on the solution of a differential equation along with initial and boundary conditions.

For their theoretical and applied importance, inverse problems for mathematical physics equations are current problems of modern mathematics. The weight of scientific-theoretical importance of inverse problems of mathematical physics in studying heat conductivity, diffusion, filtration processes in oil fields, in geophysics, quantum mechanics and biophysics caused creation of a new field of modern mathematics, theory of inverse problems of mathematical physics.

Each effective method developed to find these or other unknown characteristics of a medium simplifies and reduces the number of practical experiments and at the same time, increases their accuracy and validity.

One can find wide information on inverse problems in monographs and papers of A.Y.Akhundov, O.M. Alifanov, Y.Y. Anikonov and M.V. Neshadim, K.R. Ayda-zade and A.B. Rahimov, B.M. Budak and A.D. Iskenderov, M.I. Ismayilov, A.M. Denisov, S.J. Kabanikhin and M.A. Shishle, V.L. Kaminin, N.B. Kerimov, A.J. Kozhanov , M. Gasymov and B.M. M.Levitan, A.V. Goncharski and A.G.Yagola, M. M.Lavrentyev, A.S.Leonov, Y.T.Mehraliyev, and E.I.Azizbeyov, Q.K.Namazov, A.Zh.Prilepko and A.B. Kostin, V.Q.Romanov, Y.Q.Savatayev, A.A.Samarski and P.N.Vabishevich, A.N. Tikhonov and V.Y.Arsenin, V.K.Ivanov, Q.Y.Yagubov, J.R.Cannon and others

Object and subject of the study. Studying the well-posedness of inverse problems in Tikhonov’s sense of inverse problems on finding the unknown function in the right hand side of the equations in parabolic, hyperbolic type scalar equations, in the system of parabolic equations of special type "weak" and "reaction-diffusion" type.

Goals and objectives of the study. The goal of the work is to study uniqueness, stability and existence of the solution to inverse problems for parabolic, hyperbolic equations and reaction-diffusion type systems, in moving boundary domains for hyperbolic equation of heat conductivity and for the system of "weak" parabolic equations in cylindrical domains.

The main objective of the works to study Tikhonov well-posedness of the considered problems.

Research methods. The methods of mathematical physics, theory of differential equations and functional analysis have been used.

The main thesis to be defended.

For parabolic equations:

• Proving a theorem on the existence and stability of the solution of a Dirichlet boundary condition inverse problem on finding the right hand side in the moving boundary domain;

• Proving theorems on the existence, stability and existence of the solution to Neumann boundary condition inverse problem of finding the right hand side in the moving boundary domain;

• Proving a theorem on the uniqueness, stability of the solution to the inverse problem on finding the coefficient involved in the equation in the moving boundary domain;

For a hyperbolic equation:

• Proving theorems on the uniqueness, stability and existence of the solution to the inverse problem on finding the right hand side for hyperbolic equation of heatconductivity;

• Proving theorems on the uniqueness, stability and existence of the solution to the inverse problem on finding the right hand side of a string vibration equation in the moving boundary domain.

For the system of parabolic equations:

• Proving a theorems on the uniqueness, stability and existence of the solution of inverse problems on finding the unknown function in the right hand side in the system of "weak" parabolic equations;

• Proving a theorem on the uniqueness, stability of the solution of the inverse problem on finding the right hand side in a "reaction-diffusion" type system.

Scientific novelty of the work.

• The well-posedness of inverse problem with a coefficient (the uniqueness, stability and existence of the solution) for parabolic equations in moving boundary domains has been studied

• Theorems on the uniqueness, stability and existence of the solution to inverse problems on finding the right hand side for a string vibration equation in a moving boundary domain and hyperbolic equation of heat conductivity in a cylindrical domain has been found.
• Proving theorems on the uniqueness, stability and existence of the solution to many-dimensional inverse problems on finding the right hand side dependent on the spatial variable in the system of "weak" parabolic equations.
• Proving a theorem on the uniqueness, stability of the solution to an inverse problem on finding the right hand side dependent on time variable in "reaction-diffusion" type system.

Theoretical and practical importance of the study. The studies in the dissertation work are of theoretical character and serve to developing theory of inverse problems for mathematical physics equations.

The results of the dissertation can be used in scientific research and in working out algorithms for finding unknown physical characteristics in filtration, heat and diffusion processes.

Approbation and application of work. The main results of the work have been reported in the seminars of the department of "Mathematics and informatics" of Lankaran State University (headed by prof. A.Y. Akhundov), at the Republican scientific conference "Application of scientific innovations in teaching process" devoted to 96-th anniversary of the national leader Heydar Aliyev (Lankaran 2019), at the International scientific conference "Spectral theory and its applications" (Baku 2019), at the XXXIV International conference "Problems of decision making under uncertainties" PDMU-2019, (Ukraine, Lvov 2019), at the XXXVII International conference "Problems of decision making under uncertainties" PDMU-2022, (Sheki-Lankaran 2022), at the International conference "Modern problems of mathematics and mechanics" (Baku 2019, 2022), at the conference "Power of people, state and army" (Lankaran 2021), at the conference "Function theory, functional analysis and their applications" (Baku 2022), in the proceedings of the International conference "Modern Problems of Mathematics and Mechanics" dedicated to the 100-th anniversary of the National Leader Heydar Aliyev (Baku 2023).

Author's personal contribution. All the results of the dissertation work belong to the author.

Author's publications. The main results of the work have been published in author's 16 scientific works.

The name of the organization where the work was performed.The work was performed at Lankaran State University of the Ministry of Science and Education of the Republic of Azerbaijan,.

The structure and volume of the dissertation work (indicating, separately the volume of each structural unit in signs). The dissertation work consists of a title page -379 signs, contents - 3460 signs, introduction -37307signs, chapter I - 76000, chapter II - 62000, chapter III - 42000, conclusion -891 signs and a list of references with 91 names. The total volume of the dissertation work consists of 222037 signs.

THE CONTENT OF THE DISSERTATION WORK

The dissertation work consists of introduction, three chapters, conclusion and a list of references.

The review of the works related to the topic of the dissertation work, brief content of the results obtained in the work have been given in the introduction.

Chapter I of the work consists of three sections, chapter II and III consist of 2 sections, respectively.

Let as accept the denotations: $x=\gamma_1(t)$, $x=\gamma_2(t)$, $x=\gamma(t)$, $t\in[0,T]$, $0<T=const>0$ are the prescribed functions, $D=(\gamma_1(t),\gamma_2(t))\times(0,T]$, $B=(0,\gamma(t))\times(0,T]$, $Q_1=(0,1)\times(0,T]$, $\overline{\Omega}$ is a bounded domain of the Euclidean space R^n, $\overline{Q_n}=\overline{\Omega}\times(0,T]$, (x,t), is an arbitrary point of the domains D,B or Q_1 $(x_1,\dots,x_n,t)=(x,t)\in Q_n$.

The functional spaces $C^l(\cdot), C^{l+\alpha}(\cdot), C^{l,l/2}(\cdot), C^{l+\alpha,(l+\alpha)/2}(\cdot), l=0,1,2,\ \alpha\in(0,1)$ and the norms in these spaces are understood in the generally accepted rule:

$$\|p(x,t)\|_A^{(l)}=\sum_{j=0}^{l}\sup_A\left|\frac{\partial^j p(x,t)}{\partial x^j}\right|,\ \ \|q(t)\|_T^{(l)}=\sum_{j=0}^{l}\sup_{[0,T]}\left|\frac{d^j q(t)}{dt^j}\right|.$$

$$\|p(x,t)\|_A^{(l,k)}=\|p(x,t)\|_A^{(l)}+\|p(x,t)\|_T^{(k)},\ \upsilon=(\upsilon_1,\dots,\upsilon_m),$$

$$\|\upsilon\|_A^{(l,k)}=\sum_{i=0}^{m}\|\upsilon_i\|_A^{(l,k)}.$$

$$p_t=\frac{\partial p}{\partial t},\ p_{tt}=\frac{\partial^2 p}{\partial t^2},\ p_x=\frac{\partial p}{\partial x},\ p_{xx}=\frac{\partial^2 p}{\partial x^2},$$

$$q'=\frac{dq}{dt},\ q''=\frac{d^2 q}{dt^2},\ \upsilon_{x_i}=\frac{\partial \upsilon}{\partial x_i},\ i=\overline{1,n},$$

$$\int_A \upsilon(x,t)dx=\int_A\dots\int\upsilon(x,t)dx_1\dots dx_n,\quad \Delta\upsilon=\sum_{i=1}^{n}\frac{\partial^2\upsilon}{\partial x_i^2},\ D_x^l\upsilon$$

are l -th order acceptable derivatives of the function $\upsilon(x,t)$ with respect to x_i .

In **chapter I** we study Tikhonov well-posedness of the following inverse problems in moving boundary domains:

a) A Dirichlet boundary condition inverse problem of finding a time variable dependent unknown component in the right hand side of the parabolic equation;

b) ANeumann boundary condition inverse problem of finding the right hand side of a parabolic equation in a moving boundary domain;

c) An inverse problem of finding the unknown coefficient of a parabolic equation in a moving boundary domain.

The additional condition given for finding unknown functions in inverse problems is in the form of an integral (non-local).

Theorems on the uniqueness, stability and existence of the solution of the considered inverse problems have been proved.

In each section, examples indicating Hadamard ill-posedness of the considered inverse problem have been given.

Mixed "well-posedness" problems for parabolic equations in time dependent moving boundary domains are encountered in problems of safety of atomic energy and atomic reactors, in studying dry fuel combustion processes in rocket engines and in some other problems related to natural sciences. This time these or other characteristics of the process can not be measured directly. In this case it becomes necessary to solve inverse problems to find unknown physical characteristics by taking additional measurements.

Inverse problems on finding the unknown coefficient or the right hand side of a parabolic equation in cylindrical domains have been considered in the works of A.Y.Akhundov, O.M. Alifanov, K.R. Aydazadeh, A.M. Denisov, N.J. Ivansov, M.J. Ismayilov, V.Zh. Kaminin, N.B. Kerimov, Q.K. Namazov, A.Zh. Prilepko, Y.G. Savatayev, A.N. Tikhonov and others. Inverse problems of finding the unknown right hand side of a linear parabolic equation in moving boundary domains have been considered by A.Y. Akhundov and A.Sh. Habibova, A.Sh. Habibova (nonlocal additional condition), J.G. Malyshev (local additional condition).

In 1.1. we consider the following inverse problem of finding the pair $\{f(t), u(x,t)\}$:

Problem 1.

$$u_t - u_{xx} = f(t)g(x) \quad (x,t) \in D, \tag{1}$$

$$u(x,0)=\varphi(x),\ x\in[\gamma_1(0),\gamma_2(0)], \tag{2}$$

$$u(\gamma_1(t),t)=\psi_1(t),\ u(\gamma_2(t),t)=\psi_2(t),\ t\in[0,T], \tag{3}$$

$$\int_e^d u(x,t)dx=h(t),\quad t\in[0,T]. \tag{4}$$

For initial data of problem 1 we accept the following conditions:

1.1^0. $g(x)\in C^{\alpha}([a,b]),\ \int_e^d g(x)dx=g_0\neq 0;$

1.2^0. $\varphi(x)\in C^{2+\alpha}([\gamma_1(0),\gamma_2(0)]);$

1.3^0. $\psi_1(t),\psi_2(t)\in C^{1+\alpha}([0,T]),\ \varphi(\gamma_1(0))=\psi_1(0),\ \varphi(\gamma_2(0))=\psi_2(0);$

1.4^0. $h(x)\in C^{1+\alpha}([0,T]);$

1.5^0. $\gamma_1(t),\gamma_2(t)\in C^{1+\alpha}[0,T],\ 0<m_1\leq\gamma_2(t)-\gamma_1(t)\leq m_2<+\infty,$
$\gamma_1'(t),\gamma_2'(t)\neq 0, t\in[0,T]$ here $[a,b]$ is the projection of the domain $\overline{D}$ to the axis OX the constants e,d satisfy the condition $\gamma_1(0)<e<d<\gamma_2(0)$, m_1,m_2 are constant numbers, $\alpha\in(0,1)$.

Definition 1. The pair of functions $\{f(t),u(x,t)\}$ is said to be a classic solution of problem 1 if,

1) $f(t)\in C^{\alpha}([0,T]);$
2) $u(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{D});$
3) relations (1)-(4) for these functions are satisfied in the usual way.

In the work we give an example showing the instability of the solution of problem 1 in the sense of definition 1.

Therefore, the existence of the solution of problem 1 in any compact is accepted due to Tikhonov well-posedness principles. For problem 1 we build a set called a well-posedness class.

$$K_1^{\alpha}=\{(f,u)|\,f(t)\in C^{\alpha}([0,T]),\ u(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{D}),$$

$\exists m_3,m_4>0$ constant numbers that for $\forall\ (f,u)$ $|f(t)|\leq m_3$, $t\in[0,T]$, $|u|,|u_x|\leq m_4$, $(x,t)\in\overline{D}$ $\}$.

The uniqueness and stability of the solution is of great importance in studying the Tikhonov well-posedness of inverse problems.

Problem 1 is studied by reducing it to a problem equivalent to it. It is proved that problem 1 for finding the pair of functions $\{f(t), u(x,t)\}$ is equivalent to the problem of finding this pair of functions from the relations (1), (2), (3) and

$$f(t) = \left[h'(t) - u_x(d,t) + u_x(e,t)\right] / g_0, \ t \in [0,T], \qquad (5)$$

(by $\bar{1}$ we denote the problem of finding the pair of functions $\{f(t), u(x,t)\}$ from the relations (1), (2), (3), (5)).

We write the relations of problem $\bar{1}$ for two complect initial data $\{g_1(x), \varphi_1(x), \psi_{11}(t), \psi_{21}(t), h_1(t)\}$ (problem 1.1.) and $\{g_2(x), \varphi_2(x), \psi_{12}(t), \psi_{22}(t), h_2(t)\}$ (problem 1.2.) and denote the solutions of the obtained problems by $\{f_1(t), u_1(x,t)\}$, $\{f_2(t), u_2(x,t)\}$ respectively.

We prove the following theorem on the uniqueness and stability of the solution of problem $\bar{1}$.

Theorem 1. Assume that 1), the functions

1) $\{g_1(x), \varphi_1(x), \psi_{11}(t), \psi_{21}(t), h_1(t)\}$ and $\{g_2(x), \varphi_2(x), \psi_{12}(t), \psi_{22}(t), h_2(t)\}$ satisfy the conditions $1.1^0 - 1.4^0$ the functions $\gamma_1(t), \gamma_2(t)$ satisfy the condition 1.5^0;

2) Problem 1.1 and 1.2. have the solutions $\{f_1(t), u_1(x,t)\}$, $\{f_2(t), u_2(x,t)\}$ in the sense of definition 1 and these solutions are contained in the set K_1^{α} .

Then there exists such $T^*\left(0 < T^* \le T\right]$ that in the domain $(x,t) \in D^* = [\gamma_1(t), \gamma_2(t)] \times \left[0, T^*\right]$ the solution of problem $\bar{1}$ is unique and the stability estimations is valid:

$$\|u_1 - u_2\|_{D^*}^{0} + \|f_1 - f_2\|_{T^*}^{0} \le m\Big[\|g_1 - g_2\|_{[a,b]}^{(0)} + \|\varphi_1 - \varphi_2\|_{[\gamma_1(0), \gamma_2(0)]}^{(2)} + $$
$$+ \|\psi_{11} - \psi_{12}\|_{T^*}^{(1)} + \|\psi_{21} - \psi_{22}\|_{T^*}^{(1)} + \|h_1 - h_2\|_{T^*}^{(1)}\Big],$$

here $m > 0$ is a constant dependent on the initial data and the set K_1^{α}.

In 1.2 we study the well-posedness of an inverse problem of finding the time variable depending unknown component in the right hand side of a parabolic equation. Theorems on the uniqueness, stability and existence of the solution of the considered problem are proved.

Problem 2. On finding the following relations of the unknown pair of functions $\{f(t),u(x,t)\}$:

$$u_t - u_{xx} = f(t)g(x) \quad (x,t)\in B, \tag{6}$$

$$u(x,0)=\varphi(x), \ x\in[0,\gamma(0)], \tag{7}$$

$$u_x(0,t)=\psi_1(t), \ \ u_x(\gamma(t),t)=\psi_2(t), \ \ t\in[0,T], \tag{8}$$

$$\int_e^d u(x,t)dx = h(t), \quad t\in[0,T], \tag{9}$$

here $g(x),\varphi(x),\psi_1(t),\psi_2(t),h(t),\gamma(t)$ are the given smooth functions, $0<e<d<\gamma(0)$, e,d are constant numbers.

The solution of problem 2 is understood similar to definition 1.

It is proved that problem 2 is equivalent to the problem of finding the pair of functions $\{f(t),u(x,t)\}$ from the relations (6), (7), (8) and

$$f(t)=[h'(t)-u_x(d,t)+u_x(e,t)]/g_0, \quad t\in[0,T],$$

$$\int_e^d g(x)dx = g_0 \neq 0 \tag{10}$$

(this problem is denoted as $\overline{2}$) . For proving problem $\overline{2}$ a theorem on uniqueness and stability similar to theorem 1 is proved.

Studying Tikhonov well-posedness of ill-posed problems, usually it is taken a priori that the solution is from any compact. In some cases, it is possible to prove the existence of the solution in some sense. Alongside with completeness of studying the problem it is important from the point of view of developing algorithms for finding the solution.

Definition 2. The pair of functions $\{f(t),u(x,t)\}$ is said to be a generalized solution of problem $\overline{2}$ if 1) $f(t)\in C([0,T])$; 2) $u(x,t)\in C^{1,0}(\overline{B})$; 3) these functions satisfy the system of integral equations

$$u(x,t)=F(x,t)+y(x,t)-$$

$$-2\int_e^t G(x,t-\tau)\rho_1(\tau)d\tau+2\int_e^t G(x-\gamma(\tau),t-\tau)\rho_2(\tau)d\tau,$$

$$f(t)=[h'(t)-u_x(d,t)+u_x(e,t)]/g_0,\ t\in[0,T],$$

in the usual sense.

Here the functions $F(x,t), y(x,t), G(x,t), \rho_1(t), \rho_2(t)$ have the required smoothness and are determined as follows .

Here

$$F(x,t)=\tilde{\varphi}(x)+\frac{2\gamma(t)x-x^2}{2\gamma(t)}[\psi_1(t)-\psi_1(0)]+\frac{x^2}{2\gamma(t)}[\psi_2(t)-\psi_2(0)],$$

$$\tilde{\varphi}(x)=\begin{cases}\varphi(x), & x\in[0,\gamma(0)],\\ \varphi(\gamma(0)), & x\in\gamma(0),\ \gamma(T).\end{cases}$$

$$\phi(x,t)=f(t)g(x)+\delta_{xx}(x,t)-\delta_t(x,t)$$

$$\tilde{\phi}(x,t)=\begin{cases}\phi(0,t), & (-\infty,0)\times[0,T],\\ \phi(x,t), & [0,\gamma(t)]\times[0,T],\\ \phi(\gamma(t),t), & [\gamma(t),+\infty]\times[0,T],\end{cases}$$

$$y(x,t)=\int_0^t\int_{-\infty}^{+\infty}G(x-\xi,t-\tau)\tilde{\phi}(\xi,\tau)d\xi d\tau$$

$$G(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right),\ t>0,$$

$$\rho_1(t)=-y_x(0,t)-2\int_0^t G_x(-\gamma(t),t-\tau)\rho_2(\tau)d\tau$$

$$\rho_2(t)=-y_x(\gamma(t),t)+2\int_0^t G_x(\gamma(\tau),t-\tau)\rho_1(\tau)d\tau$$

The following theorem on the existence of generalized solution to problem $\overline{2}$ (problem 2) was proved.

Theorem 2. Assume that:

$$g(x)\in C^{\alpha}([0,\gamma(T)]),\ \int_e^d g(x)dx\neq 0,\ \varphi(x)\in C^{2+\alpha}([0,\gamma(0)]),$$

$\psi_1(t), \psi_2(t) \in C^{1+\alpha}([0,T]);\ \varphi'(0) = \psi_1(0),\ \ \varphi'(\gamma(0)) = \psi_2(0),$
$h(t) \in C^{1+\alpha}([0,T]),\ \ \gamma(t) \in C^{1+\alpha}([0,T]),\ \ \gamma'(t) > 0,\ \ t \in [0,T],$
$0 < \gamma(0) \le \gamma(t) \le \gamma(T) < +\infty$.

Then there exists such a number $T_1 (0 < T_1 < T)$ that in domain $\overline{B}^1 = [0, \gamma(t)] \times [0, T_1]$ the problem $\overline{2}$ has a solution in the sense of definition 2 .

In section 1.3 we consider an inverse problem of finding time variable dependent unknown coefficient in a parabolic equation.

Problem 3. On finding the pair of functions $\{c(t), u(x,t)\}$ from the following relations:

$$u_t - u_{xx} + c(t)u = f(x,t)\ \ (x,t) \in B, \tag{11}$$

$$u(x,0) = \varphi(x),\ \ x \in [0, \gamma(0)], \tag{12}$$

$$u_x(0,t) = \psi_1(t),\ \ u_x(\gamma(t),t) = \psi_2(t),\ \ t \in [0,T], \tag{13}$$

$$\int_e^d u(x,t)dx = h(t),\ \ \ t \in [0,T], \tag{14}$$

here $f(x,t), \varphi(x), \psi_1(t), \psi_2(t), h(t)$ are the functions with the given smoothness conditions, $0 < e < d < \gamma(0),\ e, d > 0$ are constant numbers.

A theorem on the uniqueness and stability of the solution to problem $\overline{3}$ is proved similar to theorem 1.

In **chapter II** we study the well-posedness of inverse problems of finding the unknown hand side in hyperbolic type equations.

The fundamental works of V. A. Ilin, O. A. Ladyzhenskaya, A. A. Samarski and P. N. Vabishevich, A. N. Tikhonov and A. A. Samarski are among scientific researches devoted to studying various aspects of "direct" problems. "Direct" problems for hyperbolic equation of heat conductivity have been considered in the scientific works of E.M. Kartashov and Zh.V.Antonova, V.A. Kudinov. Inverse problems with a coefficient for hyperbolic equations have been considered by Z.Aliyev and Y. T.Mehraliyev, Y.E.Anikonov and M.V.Neshadim, G.N.Isgenderova, S.Zh.Kabanikhin, A.Zh.Kozhanov, D.V.Kostin, M.A.Guliyev, A.D.Madatov, Y. T.Mehraliyev and E.Y.Azizbeyov, V.G.Romanov, A.Y.Savenkov and others.

For hyperbolic equations of heat conductivity in moving boundary domains non-local condition inverse problems for a hyperbolic equation have been considered by A.Y.Akhundov, A.Sh.Habibova.

In section 2.1. we consider an inverse problem of finding the unknown coefficient in the right hand side of a hyperbolic type heat conductivity equation. Theorems on the uniqueness, stability and existence of the considered problem have been proved.

Problem 4. We consider a problem of finding the pair of unknown functions $\{f(t), u(x,t)\}$:

$$u_t + \nu u_{tt} - u_{xx} = f(t) g(x) \quad (x,t) \in Q_1, \tag{15}$$

$$u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x), \quad x \in [0,1], \tag{16}$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{17}$$

$$\int_0^1 u(x,t)dx = h(t), \quad t \in [0,T], \tag{18}$$

here $g(x), \varphi(x), \psi(x), h(t)$ are the prescribed functions, $\nu > 0$ is a relaxation coefficient.

We accept conditions for initial data of problem 4:

2.1^0. $g(x) \in C[0,1]$, $\int_0^1 g(x)dx = g_0 \neq 0$,

2.2^0. $\varphi(x) \in C^2[0,1]$, $\psi(x) \in C^1[0,1]$, $\int_0^1 \varphi(x)dx = h(0)$, $\int_0^1 \psi(x)dx = h'(0)$;

2.3^0. $h(t) \in C^2[0,T]$.

Definition 3. The pair of functions $\{f(t), u(x,t)\}$ is said to be the solution of problem 4: 1) $f(t) \in C[0,T]$; 2) $u(x,t) \in C^{2,2}(\overline{Q_1})$, 3) for these functions the relations (15)-(18) are satisfied in a usual way.

It is proved that the problem of finding the pair of functions $\{f(t), u(x,t)\}$ from (15)-(18) (problem 4) and (15), (16), (17) (problem $\overline{4}$) from the relations

$$f(t) = [h'(t) + \nu h''(t) - u_x(1,t) + u_x(0,t)] / g_0, \quad t \in [0,T], \tag{19}$$

are equivalent.

It is assumed that for two complect initial data $\{g_1(x), \varphi_1(x), \psi_1(x), h_1(t)\}$, $\{g_2(x), \varphi_2(x), \psi_2(x), h_2(t)\}$ problem $\overline{4}$ has solutions $\{f_1(t), u_1(x,t)\}$ and $\{f_2(t), u_2(x,t)\}$ in the sense of definition 3. These problems are denoted us 4.1 and 4.2 respectively.

The set of well-posedness $K_{2,1}^{\alpha} = \{(f,u) | f(t) \in C([0,T]),\ u(x,t) \in C^{2,2}(\overline{Q_1}),\ \exists c_1, c_2 > 0$ constant numbers $\forall\ (f,u)$ that for $|f(t)| \le c_1,\quad t \in [0,T], |u|, |u_x|$ $\le c_2,\ (x,t) \in \overline{Q}_1$, for orbitrary $(f_1,u_1),(f_2,u_2) \in K_{2,1}^2$ is determined $u_{1x}(0,t) = u_{2x}(0,t), u_{1x}(1,t) = u_{2x}(1,t)\}$.

Theorem 3. Assume that

1) The functions $g_i(x), \varphi_i(x), \psi_i(x), h_i(t),\ i = 1,2$ satisfy the conditions $2.1^0 - 2.3^0$ respectively;

2) Problems 4.1 and 4.2 have classic solutions included in the set $K_{2,1}$ $\{f_1(t), u_1(x,t)\}$, $\{f_2(t), u_2(x,t)\}$.

Then there exist such a number $T^* \in (0,T]$ that in the domain $Q_1^* = [0,1] \times [0,T^*]$ the solution of problem 4 is unique and the following stability estimation is valid:

$$\int_0^1 \left[(u_{1t}(x,t) - u_{2t}(x,t))^2 + (u_{1x}(x,t) - u_{2x}(x,t))^2 \right] dx +$$

$$+ c_3 \left(\| f_1(t) - f_2(t) \|_{T^*}^{(0)} \right)^2 \le c_4 \left\{ \int_0^1 \left[(g_1(x) - g_2(x))^2 + (\varphi_1'(x) - \varphi_2'(x))^2 + \right. \right.$$

$$\left. \left. + (\psi_1(x) - \psi_2(x))^2 \right] dx + \left(\| h_1(t) - h_2(t) \|_{T^*}^{(2)} \right)^2 \right\},$$

here $c_3, c_4 > 0$ are the constants dependent on the initial data of problem 4.1 and 4.2 and the set $K_{2,1}$.

Section 2.1.3 of the work has been devoted to proving a theorem on the existence of problem $\overline{4}$.

Theorem 4. Assume that

1) $g(x) \in C^1[0,1],\ g(0) = g(1) = 0,\ \int_0^1 g(x)dx = g_0 \ne 0;$

2) $\varphi(x) \in C^2[0,1],\ \varphi(0) = \varphi(1) = 0;\ \varphi''(0) = \varphi''(1) = 0,\ \int_0^1 \varphi(x)dx = h(0);$

3) $\psi(x) \in C^1[0,1];\ \ \psi(0) = \psi(1) = 0,\ \int_0^1 \psi(x)dx = h'(0);$

4) $h(t) \in C^2[0,T].$

Then problem 4 in the domain $\overline{Q}_1 = [0,1] \times [0,T]$ has a solution in the sense of definition 3.

In 2.2. the well-posedness of the inverse problem of finding the unknown function in the right hand side of a second order hyperbolic equation in a moving boundary domain is studied.

In this section a wave equation is taken as a model.

Problem 5. On finding the pair of unknown functions $\{f(t), u(x,t)\}$ from the relations:

$$u_{tt} - u_{xx} = f(t) \cdot g(x) \quad (x,t) \in B, \tag{20}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ \ x \in [0, \gamma(0)], \tag{21}$$

$$u(0,t) = \mu_1(t),\ \ u(\gamma(t),t) = \mu_2(t), t \in [0,T], \tag{22}$$

$$\int_e^d u(x,t)dx = h(t),\ \ t \in [0,T], \tag{23}$$

here the functions $g(x), \varphi(x), \psi(x), \mu_1(t), \mu_2(t), h(t), \gamma(t)$ are the prescribed functions with certain smoothness conditions.

Theorems on the uniqueness and stability of problem $\overline{5}$ on the existence of the classic solution similar to theorem 3 and 4 have been proved.

Chapter III of the work consists of two sections. In section 1 we study the well-posedness of a nonlinear Dirichlet boundary condition many-dimensional inverse problem of finding the right hand side in the system of "weak" parabolic equations. The desired unknown functions depend on spatial variables and additional conditions for finding them are given in a non-local (integral) form.

Mixed problems for a system of parabolic equations have been studied in scientific monographs of O.A.Ladyzhenskaya, V.S.Solonnikov, N.N.Uraltseva, Dj.Marri, S.D.Eydelman and others. Inverse problems for a system of parabolic equations have been relatively little studied.

The works of A.Y.Akhundov, A.M.Denisov, A.D.Iskenderov, V.G. Yakhno, N.C. Pashayev can be shown as examples of there works.

Problem 6. We consider a problem of finding the pair of unknown functions $\left\{f_k(x), u_k(x,t), k=\overline{1,m}\right\}$:

$$u_{kt} - \Delta u_k = f_k(x) g_k(x,t), \quad (x,t) \in Q_n, \tag{24}$$

$$u_k(x,0) = \varphi_k(x), \quad x \in \overline{\Omega}, \tag{25}$$

$$u_k(x,t) = \psi_k\left(x,t,\hat{u}_k\right), \quad (x,t) \in \partial\Omega \times [0,T], \tag{26}$$

$$\int_0^T u_k(x,t)dt = r_k(x), \quad x \in \overline{\Omega}, \tag{27}$$

here $g_k(x,t), \varphi_k(x), \psi_k(x,t,\hat{p}_k), r_k(x),\ k=\overline{1,m}$ are the given smooth functions, $\hat{p}_k = \left(p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m\right)$.

Definition 4. The pairs of functions $\left\{f_k(x), u_k(x,t),\ k=\overline{1,m}\right\}$ are said to be the solutions of problem 6 if 1) $f_k(x) \in C^{\alpha}\left(\overline{\Omega}\right)$; 2) $u_k(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_n\right)$; 3) for these functions the relations (24)-(27) are satisfied in a usual way.

Let us define a set called a well-posedness class:
$K_{3,1}^{\alpha} = \left\{\left(f_k, u_k\right) \middle| f_k(x) \in C^{\alpha}\left(\overline{\Omega}\right), u_k(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_n\right), \quad \exists c_6, c_7 > 0\right.$ the constant numbers for $\forall \left(f_k, u_k\right)\ \left|f_k(x)\right| \le c_6,\ x \in \overline{\Omega}$,
$\left.\left|D_x^l u_k(x,t)\right| \le c_7,\ l=0,1,2,\ k=\overline{1,m},\ (x,t) \in \overline{Q}_n\right\}$.

Assume that the pairs $\left\{f_k^i(t), u_k^i(x,t), k=\overline{1,m}\right\}$ satisfy the relations (25)-(28) with respect to the initial data $g_k^i(\cdot), \varphi_k^i(\cdot), \psi_k^i(\cdot), r_k^i(\cdot),\ k=\overline{1,m},\ i=1,2$ (we denote these problems as problem 6.1 and 6.2 respectively).

Section 3.1.2 has been devoted to the uniqueness and stability of the solution of problem 6.

Theorem 5. Assume that

1) $g_k^i(x,t) \in C^{\alpha,\alpha/2}\left(\overline{Q}_n\right), \beta_1\sqrt{T} \le \int_0^T \left|g_k(x,t)\right| dt \le \beta_2\sqrt{T}$,

$(x,t) \in \overline{Q}_n;\ \varphi_k^i(x) \in C^{2+\alpha}\left(\overline{\Omega}\right)$

$\psi_k^i(x,t,\hat{p}_k) \in C^{\alpha,\alpha/2}\left(\overline{\Omega} \times [0,T] \times R^{m-1} = M\right),\ \psi_k^i(x,t,\hat{p}_k) - M$

$\psi_k^i(x,t,\hat{p}_k)$ satisfies the Lipshits condition with respect to the variable p in each bounded subset of the set M :

$\left|\psi_k\left(x,t,\hat{p}_k^1\right)-\psi_k\left(x,t,\hat{p}_k^2\right)\right| \le const\sum_{j=1}^{m}\left|\hat{p}_j^1-\hat{p}_j^2\right|, \left(x,t,\hat{p}_k^1\right), \left(x,t,\hat{p}_k^2\right) \in M;$

$r_k^i(x) \in C^{2+\alpha}\left(\overline{Q}\right),\ \beta_3 T \le \Delta r_k(x) \le \beta_4 T,\ k=\overline{1,m},\ i=1,2$

$\beta_1, \beta_2, \beta_3, \beta_4 > 0;$

2) Problem 6.1 and problem 6.2 have solutions $\left\{f_k^1(x), u_k^1(x,t), k=\overline{1,m}\right\}$, $\left\{f_k^{21}(x), u_k^2(x,t), k=\overline{1,m}\right\}$ contained in the set $K_{3,1}^{\alpha}$.

Then there exists such a number $T^* \times [0,T]$ that in the domain $\overline{Q}_n^* = \overline{\Omega}\times\left[0,T^*\right]$ the solution of problem 6.1 is unique and the following stability estimation is valid:

$$\left\|u^1-u^2\right\|_{Q_n}^{0} + \left\|f^1-f^2\right\|_{\overline{\Omega}}^{(0)} \le$$

$$\le c_9\left[\left\|g_k^1-g^2\right\|_{Q_n}^{0} + \left\|\varphi^1-\varphi^2\right\|_{\overline{\Omega}}^{(2)} + \left\|\psi^1-\psi^2\right\|_{M}^{(0)} + \left\|r^1-r^2\right\|_{\overline{\Omega}}^{(2)}\right],$$

here $c_9 > 0$ is a constant dependent on the initial data and the set $K_{3,1}^{\alpha}$.

Section 3.1.3 of chapter III deals with the existence of the solution of problem 6.

It is shown that if problem 6 has a solution in the set $K_{3,1}^{\alpha}$, then within the conditions imposed on the initial data problem 6 can be reduced to an equivalent problem, i.e. to a problem for a system of integral equations :

$$u_k(x,t) = \varphi_k(x) + \int_0^t\int_Q \Gamma(x,t,\xi,\tau)\left[f_k(\xi)g_k(\xi,\tau) + \Delta\varphi_k(\xi)\right]d\xi d\tau +$$

$$+\int_0^t\int_{\partial Q}\frac{\partial\Gamma(x,t,\xi,\tau)}{\partial\nu}\psi_k\left(\xi,\tau,\hat{u}_k\right)d\xi_0 d\tau,\quad k=\overline{1,m}, \tag{28}$$

$$f_k(x) = \left[u_k(x,T) - \varphi_k(x) - \Delta r_k(x)\right] \Big/ \int_0^T g_k(x,t)dt,\quad k=\overline{1,m}, \tag{29}$$

Definition 5. The pair of functions $\left\{f_k(x), u_k(x,t),\ k=\overline{1,m}\right\}$ is said to be a generalized solution to problem 6: 1) $f_k(x) \in C(\overline{\Omega})$; 2) $u_k(x,t) \in C(\overline{Q}_n)$; 3) for these functions the relations (28), (29) are satisfied in a usual way.

Theorem 6. Assume that: 1) for the initial data of problem 6 the conditions of theorem 5 are satisfied; 2) $f_k^{(0)}(x) \in C^{\alpha}(\overline{\Omega})$, $u_k^{(0)}(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_n)$, $k=\overline{1,m}$.

Then there exists such a number $T_1 (0 < T_1 \le T)$ that in the domain $\overline{Q}_n^1 = \overline{\Omega} \times [0,T_1]$ the system of integral equations (28), (29) has a solution in the sense of definition 6.

Theorem 6 is proved by the method of successive approximations.

In section 2 of chapter III we study the well-posedness of the inverse problem of finding the time variable dependent unknown function in the right hand side of a reaction diffusion type system of parabolic equations. The additional condition for the inverse problem considered in a moving boundary domain is given in the integral form.

Problem 7. We consider a problem of finding the pair of unknown functions $\{f_k(t), u_k(x,t), k=\overline{1,m}\}$:

$$u_{kt} - u_{kxx} = f_k(t) g_k(u) \quad (x,t) \in B, \tag{30}$$

$$u_k(x,0) = \varphi_k(x), \ x \in [0, \gamma(0)], \tag{31}$$

$$u_{kx}(0,t) = \psi_{1k}(t), \ u_{kx}(\gamma(t),t) = \psi_{2k}(t), \ t \in [0,T], \tag{32}$$

$$\int_e^d u_k(x,t)dx = h_k(t), \quad t \in [0,T], \tag{33}$$

here $\gamma(t), g_k(p), p = (p_1, \ldots, p_m)$, $\varphi_k(x), \psi_{1k}(x), \psi_{2k}(t), h_k(t)$ are the given smooth functions, $0 < e < d < \gamma(0)$, e, d are constant numbers, $k=\overline{1,m}$.

The solution of problem 7 is understood similar to definition 5.

Section 3.2.2 of chapter III has been devoted to proving a theorem on the uniqueness and stability of the solution to problem $\overline{7}$.

CONCLUSIONS

The dissertation work has been devoted to studyibng the well-posedness of inverse problems of finding the unknown right hand side in the system of parabolic and hyperbolic equations, in “weak” and “reaction-diffuzion” type parabolic equations. The following results have been obtained.

- The well-posedness (uniqueness, stability and existence of the solution) of inverse problems with a coefficient have been studied for parabolic equations in moving boundary domains;
- Theorems on the uniqueness, stability and existence of the solution of inverse problems of finding the right hand side for a string vibration equation in a moving boundary domain and for a heat conductivity hyperbolic equation in a cylindrical domain have been proved.
- Proving theorems on the existence, uniqueness and stability of the solution to a many-dimensional inverse problems of finding the spatial variable dependent right hand side in the system of "weak" parabolic equations.
- Proving a theorem on the uniqueness, stability of the solution to an inverse problem on finding the time variable dependent right hand side in "reaction-diffusion" type system.

The main results of the dissertation work have been published in the following works:

1. Həbibova, A.S. Hiperbolik tip istilikkeçirmə tənliyi üçün bir tərs məsələ haqqında // –Lənkəran: Lənkəran Dövlət Universitetinin Elmi Xəbərləri, Riyaziyyat və təbiət elmləri ser. -2019. №2, -s.59-63.
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