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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

ONE-SIDED GLOBAL BIFURCATION OF SOLUTIONS OF SOME NONLINEAR EIGENVALUE PROBLEMS

Specialty: 1202.01 – Analysis and functional analysis

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and development degree. The dissertation work is devoted to the study of one-sided global bifurcation of solutions from zero and infinity of nonlinear Sturm-Liouville problems with indefinite weight functions. Nonlinear eigenvalue problems for Sturm-Liouville equation with a sign-changing weight function are lately intensively studied since they arise from selection-migration models in population genetics, where a weight function represents the selective power of the environment on the genes of the population. Note that population genetics is one of the important branches of biology, which studies the genetic structure and evolution of populations. It has close ties to ecology, demography, epidemiology, phylogeny, genomics, and molecular evolution. Population genetics is mainly used in human genetics and medicine, as well as in animal and plant breeding.

Unilateral global bifurcation from zero of solutions of nonlinear Sturm-Liouville problems with a definite weight functions was studied well by P.H. Rabinowitz, A.Berestycki, K.Schmitt and H.L. Smith, J. Przybycin, B.P. Rynne, Z.S. Aliyev, G. Dai and R.Ma, Z.S. Aliyev and G.M. Mamedova. These authors have proved the existence of two families of unbounded continua of solutions in $R \times C^1$ emanating from points and intervals of the line of trivial solutions (corresponding to eigenvalues of linear Sturm-Liouville problems) and are contained in the classes of functions possessing the usual nodal properties. Note that in the works of J.F. Toland, C.A.Stuart, P.H. Rabinowitz, E.N. Dancer, I.Przybycin, B.P.Rynne, G. Dai and R. Ma have studied global bifurcation from infinity of solutions of nonlinear eigenvalue problems for the Sturm-Liouville equation with weight functions having constant signs. They have established the existence of two families of global continua of non-trivial solutions bifurcating from points and intervals of the line $R \times \{\infty\}$ corresponding to the eigenvalues of the linear Sturm-Liouville problem, and contained in the classes of functions that have usual nodal properties in some neighborhoods of these points and intervals. Similar results for fourth order nonlinear Sturm-Liouville problems with a definite weight function were obtained by Z.S. Aliyev, Z.S. Aliyev and N.A. Mustafayeva. It should be noted that when studying the bifurcation of solutions to nonlinear problems studied in the above-mentioned works, the oscillatory properties of the corresponding linear problems play a significant role.

The linear Sturm-Liouville problem with a sign-changin weight function has been studied at the beginning of the last century in the book by E.L. Ince[1], where it was proved that the spectrum of this problem consists of two positive infinitely increasing and negative infinitely decreasing sequences of simple eigenvalues. Moreover, he established that eigenfunctions corresponding to both positive and negative eigenvalues have the usual Sturm nodal properties. In addition to these properties V. Allegretto and A. Mingarelli, P.A. Binding and P.J. Brown, P.A. Binding and H. Folkmer have obtained asymptotic formulas for eigenvalues and eigenfunctions of Sturm-Liouville problems with indefinite weight functions. P. Hess and T. Kato, G.A. Afrouzi and K.J. Brown, V. Allegretto and A. Mingarelli, A. Berestycki, L. Nirenberg and S.R.S. Varadhan, K.J. Brown and S.S. Lin, K.J. Brown and A. Tertikas, J. Fleckinger and M.L. Lapidus, Y. Lu and E. Yanagida, Z.S. Aliyev and S.M. Hasanova, Z.S. Aliyev and R.A. Huseynova, R. Ma, S. Gao and X. Han have established the existence of positive and negative principal eigenvalues (i.e., eigenvalues which correspond to eigenfunctions positive in the domain) of linear eigenvalue problems for second-order elliptic partial differential equations and fourth-order ordinary differential equations with indefinite weight functions.

Global bifurcation of solutions to nonlinear linearizable eigen-value problems for second order partial elliptic differential equations with a sign-changing weight functions was studied by P. Hess and T. Kato, K.J. Brown, K.J. Brown and A. Tertikas, K.J. Brown and Y. Zhang, R.S. Cantrell and C. Cosner, M. Delgado and A. Suarez, B. Ko and K.J. Brown, Y. Su, J. Wai and J. Shi, K. Umezu, Z.S. Aliyev and Sh.M. Hasanova. They have obtained global results on bifurcation of solutions in the classes of positive functions. Global

[1]Ince, E.L. Ordinary differential equations / E.L. Ince. – London: Longmans, Green, – 1927. – 558 p.

bifurcation of solutions in the classes of positive and negative functions for fourth-order nonlinear Sturm-Liouville problems with sign-changing weight functions was studied in the works of Z.S. Aliyev and R.A. Huseynova, R. Ma, C. Gao and X. Han.

It should be noted that global bifurcation of solutions to nonlinear Sturm-Liouville problems with indefinite weight functions was studied in classes of positive functions only in the works of A. Boscaggin and F. Zanolin, M. Fencl and J. Lopez-Gomez.

Thus, the study of unilateral global bifurcation of solutions from zero and infinity of nonlinear Sturm-Liouville problems with sign-changing weight functions is relevant.

Object and subject of the study. The object of the present study is nonlinear Sturm-Liouville problems with indefine weight functions, the subject of the study is unilateral global bifurcation from zero and infinity of solutions of considered nonlinear problems.

Goal and objectives of the study. The study of unilateral global bifurcation of solutions from zero and infinity of nonlinear Sturm-Liouvulle problems with a sign-changing weight functions is the main goal and task of the given work.

Research methods. The dissertation uses methods of ordinary differential equations, spectral theories of ordinary differential operators, nonlinear functional analysis and bifurcation theory.

The main theses to be defended. The following main provisions are made for the defense of the dissertation:

– study the structure and behavior of unilateral global continua of solutions branching from zero of linearizable Sturm-Liouville problems with indefinite weight functions;

– study the structure of bifurcation points with respect to the line of trivial solutions and investigate unilateral global bifurcation of solutions of nonlinearizable eigenvalue problems for the Sturm-Liouville equation with sign-changing weight functions;

– study the structure and behavior of unilateral global continua of solutions bifurcating from infinity of asymptotically linear Sturm-Liouville problems with indefinite weights;

– study the structure of asymptotic bifurcation points and investigate the global bifurcation of solutions at infinity to nonlinearizable Sturm-Liouville problems with sign-changing weight functions.

Scientific novelty of the research. The main results of this dissertation work are the following:

– the structure and behavior of unilateral global continua of solutions branching from zero of linearizable Sturm-Liouville problems with indefinite weight functions has been completely studied;

– the structure of bifurcation points with respect to line of trivial solutions was studied, unilateral global bifurcation of solutions to nonlinearizable Sturm-Liouville problems with sign-changing weight functions was researched;

– the structure and behavior of unilateral global continua of solutions bifurcating from infinity for asymptotic linear Sturm-Liouville problems with indefinite weights is studied;

– the structure of asymptotic bifurcation points was studied and global bifurcation from infinity of solutions to nonlinearizable Sturm-Liouville problems with sign-changing weights, was studied.

Theoretical and practical value. The results obtained in the dissertation work mainly are of theoretical character. These results can be used when studying nonlinear eigenvalue problems for higher order ordinary differential equations, when modeling dynamics of population and migration-selection in population genetics.

Approbation and application. The results obtained in the dissertation work were reported at Sumgayit State University at the seminar of the department “Mathematical analysis and theory of functions” (headed by Ass. Prof. N.V. Kurbanov), at the Baku State University at the Department of Mathematical Analysis (headed by Prof. R.A. Aliyev), at the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan at the seminars of the departments “Functional Analysis” (headed by Prof. G.I. Aslanov) and “Differential Equations” (headed by Prof. A.B. Aliev), at the Republican Scientific Conference “Actual Problems of Mathematics and Mechanics” dedicated to the 94th anniversary of the National Leader of the Azerbaijani People Heydar Aliyev (BSU, Baku, 2017), at the International Scientific Conference dedicated to the 55th anniversary of Sumgait State University (SSU, Sumgait, 2017), at the International Conference “Modern Problems of Mathematics and Mechanics” dedicated to the 80th anniversary of academician A.J. Gadjiiev (IMM ANAS, Baku, 2017), at the International Conference “Modern Problems of Mathematics and Mechanics” dedicated to the 60th anniversary of the Institute of Mathematics and Mechanics (IMM ANAS, Baku, 2019), at the Republican Scientific Conference “Fundamental Problems of Mathematics and the Application of Intelligent Technologies in Education” (SSU, Sumgait, 2020), at the International Conference “Modern methods of function theory and related problems” of the Voronej Winter Mathematical School (Voronej, Russia, 2021), at the Republican conference “Current problems of mathematics and mechanics” dedicated to the 99th anniversary of the birth of the National Leader of the Azerbaijani people Heydar Aliyev (BSU, Baku, 2022).

Author’s personal contribution is in formulation of the goal of the research. Furthermore, all the results obtained belong to the author.

Authors publications. Publications in the editions recommended by HAC under President of the Republic of Azerbaijan – 5 (2 of them in WOS, 2 in SCOPUS) conference proceedings – 7 (4 in international conference, including 1 abroad, 3 in republican conferences).

The name of the organization where the dissertation work was carried out. The dissertation work was carried out at the department of “Mathematical analysis and theory of functions” of the Faculty of Mathematics of Sumgait State University.

Total volume of the dissertation work indicating separately the volume of structural units in signs.

The dissertation work consists of introduction, 2 chapters, conclusion and a list of references. The total volume of the work 201055 signs (title page 377, content 2062, introduction 49577, chapter I 88000, chapter II 60000, conclusion 1039). The list of references consists of 67 names.

THE MAIN CONTENT OF THE WORK

The dissertation work consists of introduction, II chapters, conclusion and a list of references.

In chapter I consisting of six sections we consider linearizable and nonlinearizable Sturm-Liouville problems with indefinite weight functions. Unilateral global bifurcation from zero of the solutions of these problems is studied. The existence of four families of global continua of solutions branching from the points and intervals of the line of trivial solutions and contained in the classes of functions possessing Sturm usual nodal properties, is proved.

Section 1.1 provides a statement of the problem and formulates the purpose of this chapter.

Consider the following nonlinear Sturm-Liouville problem

$$\ell(u) \equiv -(p(x)u')' + q(x)u = \lambda\rho(x)u + h\,(x,u,u',\lambda),\ 0 < x < 1, \tag{1}$$

$$\alpha_0 u(0) - \beta_0 u'(0) = 0, \tag{2}$$

$$\alpha_1 u(1) + \beta_1 u'(1) = 0, \tag{3}$$

where $\lambda \in R$ is a spectral parameter, $p(x)$ is a positive continuously differentiable function on $[0, 1]$, $q(x)$ is a non-negative continuous function on $[0, 1]$, $\rho(x)$ is a continuous sign-changing function on $[0, 1]$, $\alpha_0, \beta_0, \alpha_1, \beta_1$ are real constants such that

$$|\alpha_0| + |\beta_0| > 0,\ |\alpha_1| + |\beta_1| > 0 \text{ and } \alpha_0\beta_0 \ge 0,\ \alpha_1\beta_1 \ge 0.$$

The function h is representable in the form $h = f + g$, where the real-valued functions f and g are continuous on $[0, 1] \times R^2 \times R$ and satisfy the following conditions:

$$u\,f(x,u,s,\lambda) \le 0,\ u\,g(x,u,s,\lambda) \le 0,\ (x,u,s,\lambda) \in [0, 1] \times R^2 \times R; \tag{4}$$

there exist constants $M > 0$ and $\chi > 0$ (χ may be rather small) such that

$$\left|\frac{f(x,u,s,\lambda)}{u}\right| \le M,\ \ x \in [0,\ 1],\ (u,s) \in R^2, |u| + |s| \le \chi, \tag{5}$$

$$u \ne 0,\ \lambda \in R;$$

for each bounded interval $\Lambda \subset R$

$$g(x,u,s,\lambda) = o(|u|+|s|) \text{ as } |u|+|s| \to 0, \qquad (6)$$

uniformly with respect to $x \in [0, 1]$ and $\lambda \in \Lambda$.

In the case of

$$p \equiv 1 \text{ and } g(x,u,s,\lambda) = \lambda\rho(x)[u - m(u)],$$

where

$$m(u) = u(1-u)[h_0(1-u) + (1-h_0)u] \text{ and } h_0 \in (0, 1),$$

equation (1) is an one-dimensional reaction-diffusion equation, the interval $[0, 1]$ refers to the habitat of a species, the boundary conditions (2) and (3) for $\beta_0 = \beta_1 = 0$ means that no individuals cross the boundary of the habitat. Moreover, the weight function $\rho(x)$ represents either the selective strength of the environment on genes, or the intrinsic growth rate of the species at location x, and the real parameter λ corresponds to the reciprocal of the diffusion coefficient.

Since the condition (5) and (6) is satisfied, we consider bifurcation from $u = 0$, i.e. we study bifurcation from the line of trivial solutions. In the case of $\rho(x) > 0, x \in [0,1]$, the global bifurcation of a nonlinear eigenvalue problem (1)-(3) under conditions (5) and (6) (but without conditions (4) and $\alpha_i\beta_i \geq 0,\ i = 0, 1$) was considered in the papers of was considered in the works of Z.S. Aliev, A. Berestycki, G. Dai and R. Ma, J. Przybycin, P. Rabinowitz, B.P. Rynne, L. Schmitt and H.L. Smith. These papers it was shown the existence of two families of unbounded continua of nontrivial solutions in $R \times C^1$, possessing the usual nodal properties and bifurcating from points and intervals of the line of trivial solutions and contained in the classes of functions possessing ususal nodal properties. Similar results in nonlinear eigenvalue problems for ordinary differential equations of fourth order were established in Z.S.Aliyev’s paper.

The goal of this chapter is to study location of bifurcation points and intervals on the line of trivial solutions and structure of global continua of nontrivial solutions of problem (1)-(3), emanating from these bifurcation points and segments.

In section 1.2 we construct the classes

$$S_{k,\sigma}^{\nu},\ k\in \mathrm{N},\ \ \sigma,\nu\in\{-,+\},$$

in $R\times C^1[0,\,1]$ that play a fundamental role in studying global bifurcation of solutions of problem (1)-(3).

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(u)(x)=\lambda\rho(x)u(x),\ 0<x<1, \\ y\in B.C., \end{cases} \tag{7}$$

which obtained from (1)-(3) for $h\equiv 0$, where $B.C.$ is the set of functions satisfying boundary conditions (2) and (3).

The properties of eigenvalues and eigenfunctions of problem (7) were studied in the book of E.L. Ince[1], where in particular the following result was established.

Theorem 1. *Eigenvalues of spectral problem (7) are real, simple and consist of two unboundedly decreasing and unboundedly increasing sequences* $\{\lambda_k^-\}_{k=1}^{\infty}$ *and* $\{\lambda_k^+\}_{k=1}^{\infty}$ *respectively such that*

$$\ldots<\lambda_k^-<\ldots<\lambda_2^-<\lambda_1^-<0<\lambda_1^+<\lambda_2^+<\ \ldots<\lambda_k^+<\ldots\ .$$

Moreover, for each $k\in \mathrm{N}$ *the eigenfunctions* $u_k^-(x)$ *and* $u_k^+(x)$, *corresponding to eigenvalues* λ_k^- *and* λ_k^+, *respectively, have exactly* $k-1$ *simple zeros in the interval* $(0,1)$.

Remark 1. Since the class of continuous functions $C[0,\,1]$ is dense everywhere in $L^1[0,1]$, the above statements for problem (7) are valid for $q\in L^1[0,1]$ as well.

Lemma 1. *For each* $k\in \mathrm{N}$ *the following relations hold:*

$$\begin{aligned} &\int_0^1 \rho(x)(u_k^+(x))^2\,dx>0, \\ &\int_0^1 \rho(x)(u_k^-(x))^2\,dx<0. \end{aligned} \tag{8}$$

Let E be a Banach space $C^1[0, 1] \bigcap B.C.$ with usual norm
$$\| u \|_1 = \| u \|_\infty + \| u' \|_\infty ,$$
where
$$\| u \|_\infty = \max_{x \in [0,1]} | u(x) |.$$

In what follows on σ (respectively ν) will denote either $+$ or $-$; $-\sigma$ (respectively $-\nu$) will denote the opposite sign to σ (respectively ν).

For each $k \in \mathrm{N}$, each σ and each ν by $S_{k,\sigma}^{\nu}$ we will denote the set of functions $u \in E$, satisfying the following conditions:

(a) $u(x)$ has only simple zeros in $[0, 1]$ and has exactly $k-1$ such zeros in $(0, 1)$;

(б) $\sigma \int\limits_0^1 \rho(x) u^2(x) dx > 0$;

(в) $\lim\limits_{x \to 0+} \nu \operatorname{sgn} u(x) = 1.$

For each $k \in \mathrm{N}$ and each σ let
$$S_{k,\sigma} = S_{k,\sigma}^{-} \bigcup S_{k,\sigma}^{+}.$$
By theorem 1 and lemma 1 the sets $S_{k,\sigma}^{-}$, $S_{k,\sigma}^{+}$ и $S_{k,\sigma}$, $k \in \mathrm{N}$, are nonempty. It follows from the definition of the sets $S_{k,\sigma}^{-}$, $S_{k,\sigma}^{+}$ and $S_{k,\sigma}$, $k \in \mathrm{N}$, that they are open subsets in the space E. Note that $S_{k,\sigma}^{\nu} \bigcap S_{k',\sigma'}^{\nu'} = \varnothing$ for any $(k, \sigma, \nu) \neq (k', \sigma', \nu')$.

Lemma 2. *If* $u \in \partial S_{k,\sigma}^{\nu}$, $k \in \mathrm{N}$, *then either*

(a) there exists a point $\xi \in [0, 1]$ *such that* $u(\xi) = u'(\xi) = 0$, *or*

(б) $\int\limits_0^1 \rho(x) u^2(x) dx = 0.$

In section 1.3, problem (1)-(3) is reduced to a nonlinear operator equation with completely continuous operators (more precisely, integral operators), to which the well-known results of P. Rabinovitz[2] and E. Dancer[3] on the global bifurcation of solutions of linearizable eigenvalue problems are applicable.

In section 1.4 we study unilateral global bifurcation of solutions to problem (1)-(3) for $f \equiv 0$.

Let

$$R^+ = (0, +\infty) \text{ and } R^- = (-\infty, 0).$$

The following result holds.

Theorem 2. *For each* $k \in \mathrm{N}$ *and each* σ *there exist the continua* $C^+_{k,\sigma}$ *and* $C^-_{k,\sigma}$ *of solutionы of problems (1-(3) with* $f \equiv 0$ *that contain* $(\lambda_k^\sigma, 0)$, *are contained in*

$$(R^\sigma \times S^+_{k,\sigma}) \bigcup \{(\lambda_k^\sigma, 0)\} \text{ and } (R^\sigma \times S^-_{k,\sigma}) \bigcup \{(\lambda_k^\sigma, 0)\},$$

respectively, and are unbounded in $R^\sigma \times E$.

To prove this theorem, the following statements are required.

Lemma 3. *Let* $(\lambda, u) \in R \times E$ *be a nontrivial solution of problem (1)-(3). Then*

$$\lambda \int_0^1 \rho(x) y^2(x) dx \neq 0.$$

Lemma 4. *If* $(\lambda, u) \in R \times E$ *is a solution of problem (1)-(3) such that* $u \in \partial S^\nu_{k,\sigma}$, *then* $u \equiv 0$.

In Section 1.5 we study the properties of eigenvalues of some perturbations of the linear Sturm-Liouville problem with indefinite weight.

Along with linear spectral problem (7) we consider the following linear problem

$$\begin{cases} \ell(u)(x) + \psi(x) u(x) = \lambda \rho(x) u(x), \; x \in (0,1), \\ u \in B.C., \end{cases} \tag{9}$$

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[2]Rabinowitz, P.H. Some global results for nonlinear eigenvalue problems // J. Function. Anal., – 1971. v.7, no.3, – p. 487–513.
[3]Dancer, E.N. On the structure of solutions of nonlinear eigenvalue problems // Ind. Univ. Math. J., – 1974. v.23, no.11, – p. 1069–1076.

where

$$\psi(x) \in C[0,1] \text{ и } \psi(x) \geq 0, x \in [0,1].$$

It follows from the book of E.L. Ince[1] that eigenvalues of the linear problem (9) are real,
simple and consist of two unboundedly decreasing and unboundedly increasing sequences $\{\lambda_{k,\psi}^-\}_{k=1}^{\infty}$ and $\{\lambda_{k,\psi}^+\}_{k=1}^{\infty}$ respectively such that

$$\ldots < \lambda_{k,\psi}^- < \ldots < \lambda_{2,\psi}^- < \lambda_{1,\psi}^- < 0 < \lambda_{1,\psi}^+ < \lambda_{2,\psi}^+ < \ldots < \lambda_{k,\psi}^+ < \ldots .$$

Let

$$M_\psi = \sup_{x \in [0,1]} \psi(x).$$

Next, by λ_{k,M_ψ}^+ and λ_{k,M_ψ}^-, $k \in \mathrm{N}$, we denote kth positive and negative eigenvalues of the spectral problem

$$\begin{cases} \ell(u)(x) + M_\psi u(x) = \lambda \rho(x) u(x), \ x \in (0,1), \\ u \in B.C. \end{cases} \tag{10}$$

To study the global bifurcation of solutions to problem (1)-(3), we need the following statements.

Lemma 5. *For each* $k \in \mathrm{N}$ *the following relations hold:*

$$\begin{aligned} &\lambda_k^+ \leq \lambda_{k,\psi}^+ \leq \lambda_{k,M_\psi}^+, \\ &\lambda_{k,M_\psi}^- \leq \lambda_{k,\psi}^- \leq \lambda_k^-. \end{aligned} \tag{11}$$

Lemma 6. *The following relations hold:*

$$\lambda_{1,\psi}^+ - \lambda_1^+ \leq \frac{M_\psi \int_0^1 (u_1^+(x))^2 \, dx}{\int_0^1 \rho(x) \, (u_1^+(x))^2 \, dx},$$

$$\lambda_{1,\psi}^- - \lambda_1^- \leq - \frac{M_\psi \int_0^1 (u_1^-(x))^2 \, dx}{\int_0^1 \rho(x) \, (u_1^-(x))^2 \, dx}.$$

Remark 2. Based on Remark 1, the statements of Lemmas 5 and 6 are also true in the case $\psi \in L^1[0, 1]$.

In 1.6, unilateral global bifurcation of solutions to problem (1)-(3) is studied in the case when the function f is not identical zero. At the same time, the bifurcation of solutions to problem (1)-(3) from intervals of the line of trivial solutions is studied. Recall that if an interval contains at least one bifurcation point, then this intervalis called a bifurcation interval.

Let $(\lambda, 0)$ be a bifurcation point of problem (1)-(3). If there exists a sequence

$$\{(\lambda_{n,k,\sigma}^{\nu}, u_{n,k,\sigma}^{\nu})\}_{n=1}^{\infty} \subset (C \bigcap (R^{\sigma} \times S_{k,\sigma}^{\nu}))$$

such that

$$\lambda_{n,k,\sigma}^{\nu} \to \lambda, \ \| u_{n,k,\varsigma}^{\nu} \|_1 \to 0 \ \text{ as } n \to +\infty,$$

then $(\lambda, 0)$ is said to be a bifurcation point of problem (1)-(3) with respect to the set $R^{\sigma} \times S_{k,\sigma}^{\nu}$.

We introduce the following notations:

$$d_1^{\sigma} = \frac{M \int_0^1 (u_1^{\sigma}(x))^2 \, dx}{\int_0^1 \rho(x) \, (u_1^{\sigma}(x))^2 \, dx}, \ \sigma \in \{+, -\}.$$

$$I_1^+ = [\lambda_1^+, \lambda_1^+ + d_1^+], \quad I_1^- = [\lambda_1^- - d_1^-, \lambda_1^-],$$

$$I_1^+(\delta) = [\lambda_1^+ - \delta, \lambda_1^+ + d_1^+ + \delta], \quad I_1^-(\delta) = [\lambda_1^- - d_1^- - \delta, \lambda_1^- + \delta],$$

$$I_k^+ = [\lambda_k^+, \lambda_{k,M}^+], \quad I_k^- = [\lambda_{k,M}^-, \lambda_k^-],$$

$$I_k^+(\delta) = [\lambda_k^+ - \delta, \lambda_{k,M}^+ + \delta], I_k^-(\delta) = [\lambda_{k,M}^- - \delta, \lambda_k^- + \delta]$$

for $k \geq 2,$ where δ is some positive number.

Lemma 7. *For each* $k \in \mathrm{N}$, *each* σ, *each* ν *and each sufficiently small* $r \in (0, \chi)$ *there exist a solution* $(\lambda_{k,\sigma,r}^{\nu}, u_{k,\sigma,r}^{\nu})$ *of* *problem (1)-(3) and the number* $\delta_{k,\sigma}^{\nu} > 0$ *such that*

$$\lambda_{k,\sigma,r} \in I_k^{\sigma}(\delta_{k,\sigma}^{\nu}),\ \|u_{k,\sigma,r}^{\nu}\|_1 = r \text{ and } u_{k,\sigma,r}^{\nu} \in S_{k,\sigma}^{\nu}.$$

Corollary 1. *For each* $k \in \mathrm{N}$, *each* σ *and each* ν *the set of bifurcation points of problem (1)-(3) with respect to the set* $R^{\sigma} \times S_{k,\sigma}^{\nu}$ *is non-empty.*

Lemma 8. *Let* $(\lambda, 0)$ *be a bifurcation point of problem (1)-(3) with respect to the set* $R^{\sigma} \times S_{k,\sigma}^{\nu}$. *Then* $\lambda \in I_k^{\sigma}$.

Let D be a closure in $R \times E$ of the set of nontrivial solutions of problem (1)-(3). For each $k \in \mathrm{N}$, each σ and each ν by $\tilde{D}_{k,\sigma}^{\nu}$ we denote the union of all connected components $D_{k,\sigma,\lambda}^{\nu}$ of the set D, that are emanating from bifurcation points $(\lambda, 0) \in I_k^{\sigma} \times \{0\}$ with respect to the set $R^{\sigma} \times S_{k,\sigma}^{\nu}$. Note that

$$D_{k,\sigma}^{\nu} = \tilde{D}_{k,\sigma}^{\nu} \cup (I_k^{\sigma} \times \{0\})$$

is a connected subset of $R \times E$, but $\tilde{D}_{k,\sigma}^{\nu}$ need not to be connected in $R \times E$.

The following theorem is the main result of this chapter.

Theorem 3. *For each* $k \in \mathrm{N}$ *and each* σ *the sets* $D_{k,\sigma}^{+}$ *and* $D_{k,\sigma}^{-}$ *are contained in*

$$(R^{\sigma} \times S_{k,\sigma}^{+}) \cup (I_k^{\sigma} \times \{0\}) \text{ and } (R^{\sigma} \times S_{k,\sigma}^{-}) \cup (I_k^{\sigma} \times \{0\})$$

respectively and are unbounded in $R \times E$.

In chapter II we study unilateral global bifurcation of linearizable and nonlinearizable at infinity nonlinear Sturm-Liouville problems with a sign-changing weight functions. The existence of four families of unbounded continua of solutions bifurcating from points and intervals of the line $R \times \{\infty\}$ and contained in the classes $R^{\sigma} \times S_{k,\sigma}^{\nu}$ in some neighborhoods of these points and intervals is proved.

In 2.1 we present the formulation of the bifurcation problem from infinity in nonlinear Sturm-Liouville problems with an indefinite weight function. Here we continue the study of the nonlinear problem (1)-(3) in the case when the functions f and g, along with condition (4), also satisfy the following conditions: there exists a constant $M>0$ such that

$$\left|\frac{f(x,u,s,\lambda)}{u}\right| \le M,\ x\in[0,1],\ (u,s)\in R^2,\ \ u\ne 0,\ \lambda\in R; \quad (12)$$

for each bounded interval $\Lambda\subset R$

$$g(x,u,s,\lambda)=o(|u|+|s|) \text{ as } |u|+|s|\to\infty, \quad (13)$$

uniformly with respect to $x\in[0,1]$ and $\lambda\in\Lambda.$

In 2.2 we study unilateral global bifurcation from infinity of solutions to problem (1)-(3) for $f\equiv 0.$

Remark 3. We add the points of infinity $\{(\lambda,\infty):\lambda\in R\}$ to the space $R\times E$ and determine suitable topology on the resulting set. Then for each $\lambda\in R$ the point (λ,∞): will be an element of our space $R\times E.$

By $\hat{D}$ we denote the set of nontrivial solutions of (1)-(3).

The following theorem is one of the main results of this chapter.

Theorem 4. *Let* $f\equiv 0$ *in equation (1). Then for each* $k\in\mathrm{N}$ *and each* σ *there exist the connected components* $\hat{C}^{+}_{k,\sigma}$ *and* $\hat{C}^{-}_{k,\sigma}$ *of the set* $\hat{D}$ *and the neighborhood* $\hat{Q}_{k,\sigma}$ *of the point* $(\lambda_k^{\sigma},\infty)$ *such that*

(a) $\hat{C}^{+}_{k,\sigma}\cap\hat{Q}_{k,\sigma}\subset R^{\sigma}\times S^{+}_{k,\sigma}$ *and* $\hat{C}^{-}_{k,\sigma}\cap\hat{Q}_{k,\sigma}\subset R^{\sigma}\times S^{-}_{k,\sigma}$;

(b) one of the following statements hold:

(b1) $\hat{C}^{\nu}_{k,\sigma}\setminus Q_{k,\sigma}$ *intersects the point* $(\lambda_{k'}^{\sigma},\infty)$ *with respect to the set* $R^{\sigma}\times S^{\nu'}_{k',\sigma}$ *for some* $(k',\nu')\ne(k,\nu)$;

(b2) $\hat{C}^{\nu}_{k,\sigma}\setminus Q_{k,\sigma}$ *intersects* R_0 *for some* $\lambda\in R$;

(b3) the projection of the set $\hat{C}^{\nu}_{k,\sigma}\setminus Q_{k,\sigma}$ *on* R_0 *is unbounded, where* $R_0=\{(\lambda,0):\lambda\in R\}.$

In 2.3 we prove the existence of asymptotic bifurcation points of the problem (1)-(3) with respect to the set $R^{\sigma} \times S_{k,\sigma}^{\nu}$.

Lemma 9. *For each* $k \in \mathrm{N}$, *each* σ, *each* ν *and for sufficiently large* $R > 0$ *there exists a solution* $(\lambda_{k,\sigma,R}^{\nu}, u_{k,\sigma,R}^{\nu})$ *of problem (1)-(3) such that*

$$\lambda_{k,\sigma,R}^{\nu} \in R^{\sigma},\ u_{k,\sigma,R}^{\nu} \in S_{k,\sigma}^{\nu} \ \textit{ and } \ \| u_{k,\sigma,R}^{\nu} \|_1 = R.$$

Let τ_0 be an arbitrary fixed sufficiently small positive number.

Corollary 2. *For each* $k \in \mathrm{N}$, *each* σ *and each* ν *there exists a sufficiently large positive number* $R_{k,\sigma}^{\nu}$ *such that for any* $R \geq R_{k,\sigma}^{\nu}$ *problem (1)-(3) has a solution* (λ, u) *which satisfies the following conditions:*

$$\lambda \in I_k^{\sigma}(\tau_0),\ u \in S_{k,\sigma}^{\nu} \ \textit{ and } \ \| u \|_1 = R.$$

Recall that the point (λ, ∞), $\lambda \in R^{\sigma}$, is an asymptotic bifurcation point of problem (1)-(3) with respect to the set $R^{\sigma} \times S_{k,\sigma}^{\nu}$, if for any sufficienly small $r > 0$ there exists a solution $(\lambda_{k,\sigma,r}^{\nu}, u_{k,\sigma,r}^{\nu})$ such that

$$| \lambda - \lambda_k^{\sigma} | < r, \quad \| u \|_1 > \frac{1}{r} \quad \text{and} \quad u \in S_{k,\sigma}^{\nu}.$$

By virtue of Lemma 9 and Corollary 2, the following corollary holds.

Corollary 3. *For each* $k \in \mathrm{N}$, *each* σ *and each* ν *the set of asymptotic bifurcation points of problem (1)-(3) with respect to the set* $R^{\sigma} \times S_{k,\sigma}^{\nu}$ *is nonempty Moreover, if* $(\lambda, 0)$ *is an aasymptotic bifurcation point of problem (1)-(3) with respect to the set* $R^{\sigma} \times S_{k,\sigma}^{\nu}$, *then* $\lambda \in I_k^{\sigma}$.

Section 2.4 studies the structure of unilateral global continua emanating from the asymptotic bifurcation points of problem (1)-(3) in the case when the function f is not identically zero.

For each σ let

$$R_0^\sigma = \{(\lambda, 0) : \lambda \in R^\sigma\}$$

and

$$R_\infty^\sigma = \{(\lambda, \infty) : \lambda \in R^\sigma\}.$$

For each $k \in \mathrm{N}$, each σ and each ν we determine the set $\hat{D}_{k,\sigma}^{\nu,*}$ as the union of all components of $\hat{D}$, bifurcating from $I_k^\infty \times \{\infty\}$ with respect to the set $R^\sigma \times S_{k,\sigma}^\nu$. Moreover, let

$$\hat{D}_{k,\sigma}^\nu = \hat{D}_{k,\sigma}^{\nu,*} \bigcup (I_k^\sigma \times \{\infty\}).$$

The following theorem is the main result of this chapter.

Theorem 5. *For each $k \in \mathrm{N}$, each σ and each ν the set $\hat{D}_{k,\sigma}^\nu$ is contained in $R^\sigma \times E$, and for this set one of the following statements holds:*

(a) there exists $(k', \nu') \neq (k, \nu)$ such that $\hat{D}_{k,\sigma}^\nu$ intersects $I_{k'}^\sigma \times \{\infty\}$ with respect to the set $R^\sigma \times S_{k',\sigma}^{\nu'}$;

(b) there exists $\lambda \in R^\sigma$ such that $\hat{D}_{k,\sigma}^\nu$ intersects R_0^σ at the point $(\lambda, 0)$;

(c) the projection $pr_{R_0^\sigma} \hat{D}_{k,\sigma}^\nu$ of the set $\hat{D}_{k,\sigma}^\nu$ on R_0^σ is unbounded.

In section 2.5 we study global bifurcation of solutions to problem (1)-(3) subject to the conditions (4), (6), (12) and (13), where theorem 3 and 5 are improved as follows.

Theorem 6. *Let the conditions (4), (6), (12) and (13) be satisfied. Then for each $k \in \mathrm{N}$, each σ and each ν the set $\hat{D}_{k,\sigma}^\nu$ be contained in $(R^\sigma \times S_{k,\sigma}^\nu) \bigcup (I_k^\sigma \times \{\infty\})$, and consequently the alternative (a) of theorem 5 does not hold. Moreover, if $\hat{D}_{k,\sigma}^\nu$ intersects R_0^σ for some $\lambda \in R^\sigma$, then $\lambda \in I_k^\sigma$, and if $\tilde{D}_{k,\sigma}^\nu$ intersects R_∞^σ for some $\lambda \in R^\sigma$, then $\lambda \in I_k^\sigma$.*

CONCLUSION

The dissertation work considers nonlinear Sturm-Liouville problems with indefinite weight functions. Note that problems of this type arise when modeling selection-migration in population genetics. Unilateral global bifurcation of solutions from zero and infinity of these nonlinear eigenvalue problems are studied.

The following results are the main ones for this dissertation:

– the structure and behavior of unilateral global continua of solutions branching from zero of linearizable Sturm-Liouville problems with indefinite weight functions has been completely studied;

– the structure of bifurcation points with respect to line of trivial solutions was studied, unilateral global bifurcation of solutions to nonlinearizable Sturm-Liouville problems with sign-changing weight functions was researched;

– the structure and behavior of unilateral global continua of solutions bifurcating from infinity for asymptotic linear Sturm-Liouville problems with indefinite weights is studied;

– the structure of asymptotic bifurcation points was studied and global bifurcation from infinity of solutions to nonlinearizable Sturm-Liouville problems with sign-changing weights, was studied.

The main results of the dissertation work were published in the following works:

1. Nəsirova, L.V. İndefinit çəkili Sturm-Liuvill məsələsinin məxsusi ədədlərinin həyəcanlanmaları // "Riyaziyyatın fundamental problemləri və intellektual texnologiyaların təhsildə tətbiqi" adlı Respublika elmi konfransının materialları, – Sumqayıt: – 2020, – s. 51-52.

2. Алиев, З.С., Ашурова, Л.В. О бифуркации решений нелинейной задачи Штурма-Лиувилля с индефинитным весом // Материалы международной научной конференции посвященной 55-летному юбилею Сумгаитского Государственного Университета, – Сумгаит: – 2017, – s. 56-57.

3. Насирова, Л.В. Глобальная бифуркация решений нелинеаризуемой задачи Штурма-Лиувилля с индефенитным весом // Azərbaycan xalqının Ümummilli lideri Heydər Əliyevin anadan olmasının 94-illik yubileyinə həsr olunmuş "Riyaziyyat və mexanikanın aktual problemləri" adlı respublika elmi konfransı, Bakı: – 2017, – s. 160–162.

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